M-QUASI-HYponORMAL COMPOSITION OPERATORS

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ABSTRACT. A necessary and sufficient condition is obtained for M-quasi-hyponormal composition operators. It has also been proved that the class of M-quasi-hyponormal composition operators coincides with the class of M-paranormal composition operators. Existence of M-hyponormal composition operators which are not hyponormal; and M-quasi-hyponormal composition operators which are not M-hyponormal and quasi-hyponormal are also shown.

KEY WORDS AND PHRASES. M-hyponormal, M-quasi-hyponormal, M-paranormal, normal composition operators.

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1. PRELIMINARIES.

Let \( (X, S, m) \) be a sigma-finite measure space and \( T \) a measurable transformation from \( X \) into itself (that is one \( m(T^{-1}(E)) = 0 \) whenever \( m(E) = 0 \) for \( E \in S \)). Then the equation \( C_T f = f \circ T \) for every \( f \) in \( L^2(m) \) defines a linear transformation. If \( C_T \) is bounded with range in \( L^2(m) \), then it is called composition operator. If \( X = \mathbb{N} \) the set of all non-zero positive integers and \( m \) is counting measure on the family of all subsets of \( \mathbb{N} \), then \( L^2(m) = l^2 \) (the Hilbert space of all square summable sequences).

Let \( f_o = \frac{dmT^{-1}}{dm} \) be the Radon-Nikodym derivative of the measure \( mT^{-1} \) with respect to the measure \( m \),

\[
\frac{dm(ToT)^{-1}}{dmT^{-1}} = g_o, \quad \frac{dm(ToT)^{-1}}{dm} = h_o
\]

Then \( h_o = f_o \cdot g_o \).

Let \( B(H) \) denote the Banach algebra of all bounded linear operators on the Hilbert space \( H \). An operator \( T \in B(H) \) is called M-quasi-hyponormal if there exists \( M > 0 \) such that

\[
M^2 T^* (T^2 - (TT)^2) \geq 0
\]
or equivalently \(||T^2x|| \leq M||T^2x||\) for all \(x\) in \(H\). \(T\) is said to be M-paranormal [2] if for all unit vectors \(x\) in \(H\)

\[ ||Tx||^2 \leq M||T^2x||. \]

\(T\) is said to be M-hyponormal [2] if

\[ ||Tx|| \leq M||Tx|| \] for all \(x\) in \(H\).

The purpose of this paper is to generalize the results on quasi-hyponormal composition operators in [3] for M-quasi-hyponormal composition operators.

2. M-QUASI-HYPONORMAL COMPOSITION OPERATORS.

In this section we obtain a necessary and sufficient condition for M-quasi-hyponormal composition operators and then show that the class of M-quasi-hyponormal composition operators on \(L^2\) coincides with the class of M-paranormal composition operators. We also show the existence of M-hyponormal composition operators which are not hyponormal, and M-quasi-hyponormal composition operators which are not M-hyponormal and quasi-hyponormal.

**Theorem 2.1.** Let \(C_T \in B(L^2)\). Then \(C_T\) is M-quasi-hyponormal if and only if

\[ f^2 \leq M^2 h_0. \]

**Proof.** Since for any \(f\) in \(L^2\),

\[ (C_T^2 C_T^* f, f) = (C_T^* C_T f, f) = \int h_0 |f|^2 \, dm, \]

\[ = (M h_0 f, f), \]

where \(M h_0\) is the multiplication operator induced by \(h_0\), therefore \(C_T^2 C_T^* = M h_0\).

Similarly, it can be seen that \(C_T^* C_T = M f_0\). \(C_T\) is M-quasi-hyponormal if and only if

\[ M^2 C_T^2 C_T^* - (C_T^* C_T)^2 \geq 0. \]

This implies that

\[ M^2 M h_0 - M^2 f_0 \geq 0, \]

that is \(f_0^2 \leq M^2 h_0\).

Hence the result.

**Corollary.** Let \(C_T \in B(L^2)\). Then \(C_T\) is M-quasi-hyponormal if and only if

\[ f_0 \leq M^2 g_0. \]

**Proof.** Since \(h_0 = f_0 g_0\) and \(f_0\) is positive, therefore, by above theorem we get the result.

**Theorem 2.2.** Let \(C_T \in B(L^2)\). Then \(C_T\) is M-quasi-hyponormal if and only if \(C_T\) is M-paranormal.

**Proof.** Necessity is true for any bounded operator \(A\). For the sufficiency, let \(C_T\) be M-paranormal, then
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\[ ||C_T X_{\{n\}}||^2 \leq M ||C_T^2 X_{\{n\}}|| \quad \text{for all } n \in \mathbb{N} \]

or

\[ \int |X_{\{n\}} \circ T|^2 dm \leq M \left( \int |X_{\{n\}} \circ T^2|^2 dm \right)^{1/2} \]

or

\[ \int |X_{\{n\}}|^2 dm T^{-1} \leq M \left( \int |X_{\{n\}}|^2 dm (T^o T)^{-1} \right)^{1/2} \]

or

\[ \int f_{\{n\}} \circ o dm \leq M \left( \int f_{\{n\}} \circ o^2 \right)^{1/2} \]

or

\[ f^2_{\circ o}(n) \leq M^2 h_{\circ o}(n) \quad \text{for all } n \in \mathbb{N}. \]

Hence \( f^2_{\circ o} \leq M^2 h_{\circ o} \); \( C_T \) is M-quasi-hyponormal.

THEOREM 2.3. Let \( C_T \in B(z^2) \) and \( T: \mathbb{N} \to \mathbb{N} \) be one-to-one. Then the following are equivalent.

(i) Normal
(ii) M-hyponormal
(iii) M-quasi-hyponormal.

PROOF. (i) implies (ii), (ii) implies (iii) are always true for any bounded operator \( A \). We show that (iii) implies (i). Let \( C_T \) be M-quasi-hyponormal. Then

\[ ||C_T^* C_T f|| \leq M ||C_T f|| \quad \text{for all } f \in z^2. \]

Now \( T \) is onto because if \( T \) is not onto then \( N|T(N)| \) is non-empty and for \( n \in \mathbb{N} \)

\[ ||C_T X_{\{n\}}|| = 1 \quad \text{and} \quad ||C_T^* X_{\{n\}}|| = 0. \]

There exists no \( M>0 \) such that \( C_T \) is M-quasi-hyponormal which is a contradiction.

Since \( T \) is one-to-one, therefore, \( T \) is invertible, by Theorem 2.2 [4] \( C_T \) is invertible and \( C_T \) is normal by Theorem 2.1 [3].

Here we give an example of a composition operator on \( z^2 \) which is M-hyponormal but not hyponormal.

EXAMPLE 1. Let \( T: \mathbb{N} \to \mathbb{N} \) be the mapping such that

\[ T(1) = 2, \quad T(2) = 1, \quad T(3) = 2 \]

\[ T(3n+m) = n+2, \quad m = 1,2,3 \quad \text{and} \quad n \in \mathbb{N}. \]

Then \( C_T \) is not hyponormal as \( f_{\circ o} \neq g_{\circ o} \) for \( n = 1. \) \( C_T \) is M-hyponormal for \( M \geq \sqrt{2}. \)

EXAMPLE 2. Let \( T: \mathbb{N} \to \mathbb{N} \) be defined by \( T(1) = 2, \ T(2) = 1, \ T(3n+m) = n+1, \)

\[ m = 0,1,2 \quad \text{and} \quad n \in \mathbb{N}. \]

Then \( C_T \) is \( \sqrt{2} \) - quasi-hyponormal but \( C_T \) is not \( \sqrt{2} \)

-hyponormal. \( C_T \) is not quasi-hyponormal also.

REFERENCES


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