Internat. J. Math. & Math. Sci.
Vol. 10, No. 3 (1987) 551-556

TOTALLY UMBILICAL CR-SUBMANIFOLDS OF SEMI-RIEMANNIAN KAEHLER MANIFOLDS

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(Received August 18, 1986 and in revised form October 20, 1986)

ABSTRACT. We study totally umbilical CR-submanifolds of a Kaehler manifold carrying a semi-Riemannian metric. It is shown that for dimension of the totally real distribution greater than one, these submanifolds are locally decomposable into a complex and a totally real submanifold of the Kaehler manifold. For dimension equal to one, we show, in particular, that they are endowed with a normal contact metric structure if and only if the second fundamental form is parallel.

KEY WORDS AND PHRASES. CR-submanifolds, Totally umbilical, Semi-Riemannian metric, Kaehler manifold, normal contact metric structure, second fundamental form.
1980 AMS SUBJECT CLASSIFICATION CODE. 53C40.

1. INTRODUCTION.

The notion of a Cauchy-Riemann (CR)-submanifold of a Kaehler manifold was introduced by Bejancu [1,2] and developed later by several researchers. For a detailed treatment, we refer to Yano and Kon [3]. The study on CR-submanifolds has so far been confined to a positive definite metric. The objective of this paper is to study totally umbilical CR-submanifolds of a Kaehler manifold carrying a semi-Riemannian metric. According to Flaherty's theorem [4], the Hermitian metric on a Kaehler manifold cannot have the Lorentzian signature (which is essential for a space-time manifold of relativity). This motivates us to consider CR-submanifolds (whose metric can have the Lorentzian signature) of a Kaehler manifold.

2. PRELIMINARIES.

Let M be a submanifold isometrically immersed in a Kaehler manifold $\bar{M}$ with a complex structure J, a Hermitian metric g and the Levi-Civita connection $\bar{\nabla}$. We denote by the same g the metric of M and $\bar{M}$, which is assumed to be semi-Riemannian. We also assume that TM is non-degenerate with respect to g. If ∇ is the Levi-Civita connection induced in M then we have

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y) \qquad \text{(Gauss Formula)} \tag{2.1}$$

$$\bar{\nabla}_X V = -A_V X + D_X V \qquad \text{(Weingarten Formula)} \tag{2.2}$$

for arbitrary vector fields X,Y tangential to M and V normal to M; where B is the second fundamental form, A_V the shape operator associated to V and D the normal connection of M. In fact, $g(A_V X,Y) = g(B(X,Y),V)$. M is said to be totally umbilical if $B(X,Y) = g(X,Y)\mu$, μ being the mean curvature vector field of M. In particular, when $B = 0$, we say that M is totally geodesic.

M is said to be a CR-submanifold [1] of the Kaehler manifold $\bar{M}$ if there exists a differentiable distribution D on M such that (i) D is invariant by J, i.e. $JD_x = D_x$ for each x in M and (ii) the complementary orthogonal distribution $D^\perp$ is anti-variant (totally real), i.e. JD_x is a subspace of $T_x(M)^\perp$ for each x in M. A CR-manifold is called proper if $\dim D \neq 0$ and $\dim D^\perp \neq 0$. Let

$$JX = PX + FX, \qquad JV = tV + fV \tag{2.3}$$

be the decompositions of JX and JV into their tangential and normal components respectively. Following relations hold [3]:

$$P^3 + P = 0, \qquad f^3 + f = 0 \tag{2.4a}$$

$$FP = 0, \quad fF = 0, \quad tf = 0, \quad Pt = 0 \tag{2.4b}$$

$$(\nabla_X P)Y = A_{FY}X + yB(X,Y) \tag{2.5}$$

$$(\nabla_X F)Y = -B(X,PY) + fB(X,Y) \tag{2.6}$$

$$(\nabla_X t)V = A_{fV}X - PA_V X \tag{2.7}$$

$$(\nabla_X f)V = -FA_V X - B(X,tV) \tag{2.8}$$

where $(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y$, $(\nabla_X F)Y = D_X(FY) - F\nabla_X Y$

$$(\nabla_X t)V = \nabla_X(tV) - tD_X V, \quad (\nabla_X f)V = D_X(fV) - fD_X V$$

PROPOSITION 2.1. Let M be a CR-submanifold of a Kaehler manifold. Then both the distributions D and $D^\perp$ are non-degenerate.

PROOF. Let D be degenerate. Then there exists a non-zero vector field X in D such that $g(X,Y) = 0$ for all Y in D. As D and $D^\perp$ are complementary and orthogonal to each other, it follows that $g(X,Y) = 0$ for all Y in TM. This shows that $X = 0$ because TM is non-degenerate. But X is non-zero. Hence we arrive at a contradiction. This proves non-degenerateness of D. That $D^\perp$ is non-degenerate can be proved likewise.

PROPOSITION 2.2. The mean curvature vector μ of a totally umbilical CR-submanifold of a Kaehler manifold belongs to $JD^\perp$.

PROOF. Consider any X in D and V in the complementary orthogonal subbundle to $JD^\perp$ in $TM^\perp$. Then we have

$$g(J(\bar{\nabla}_X X),JV) = g(\bar{\nabla}_X JX,JV) = g(\nabla_X JX + g(X,JX)\mu,JV) = 0.$$

$$g(J(\bar{\nabla}_X X),JV) = g(\bar{\nabla}_X X,V) = g(\nabla_X X + g(X,X)\mu,V) = g(X,X)g(\mu,V).$$

Thus we observe that $g(X,X)g(\mu,V) = 0$. By prop. 2.1 it follows that $g(\mu,V) = 0$; i.e. $f\mu = 0$. Hence μ belongs to $JD^\perp$.

LEMMA 2.1. (Yano-Kon [3]) Let M be a CR-manifold of a Kaehler manifold. Then $A_{FX}Y = A_{FY}X$, for any X,Y in $D^{\perp}$.

The above lemma holds for both positive definite and indefinite metrics. Let us denote $\dim D^{\perp}$ by q.

3. TOTALLY UMBILICAL CR-SUBMANIFOLDS WITH $q > 1$.

For a positive definite metric, the following is known:

THEOREM 3.1. (Bejancu [5]). Let M be a totally umbilical proper CR-submanifold of a Kaehler manifold $\bar{M}$. If $q > 1$, then M is totally geodesic in $\bar{M}$ and is locally a product of the leaves of D and $D^{\perp}$, i.e. a CR-product.

That the above theorem holds also for an indefinite metric, can be proved as follows.

PROOF. By lemma 2.1, we have $A_{FX}Y = A_{FY}X$ for all X,Y in $D^{\perp}$. As $t\mu$ lies in $D^{\perp}$, for any X in $D^{\perp}$ we have $A_{FX}t\mu = A_{Ft}\mu X$. Since M is totally umbilical we have $B(S,Y) = g(X,Y)\mu$ and $A_V X = g(\mu,V)X$. Hence we obtain

$$g(t\mu,X)t\mu = g(t\mu,t\mu)X \tag{3.1}$$

for all X in $D^{\perp}$. Since $q > 1$, it follows, on contraction of (3.1) at X with respect to an orthonormal base of $D^{\perp}$, that $g(t\mu,t\mu) = 0$. Thus we get $t\mu = 0$.

Now, by prop. 2.2 we also have $f\mu = 0$. Consequently $J\mu = 0$ and hence $\mu = 0$. M therefore reduces to a totally geodesic submanifold. Obviously we have $\nabla P = 0$. Thus, according to Chen's theorem [6], "A CR-submanifold is a CR-product iff. $\nabla P = 0$", it follows that M is a CR-product. This completes the proof.

REMARK 1. If we take $\dim D = \dim D^{\perp} = 2$ so that $\dim M = 4$ and assume that the f-structure $P^3 + P = 0$ on M is globally framed, then there exists a local basis $\{\xi_1,\xi_2,\xi_3,\xi_4\}$ for M such that $P\xi_1 = \xi_2$, $P\xi_2 = -\xi_1$, $P\xi_3 = P\xi_4 = 0$. One can verify that g is Lorentzian with signature $+ + + -$. Subject to the hypothesis of theorem 3.1, M serves as a model of a class of decomposable space-times in general relativity. For a recent study on such space-times we refer to [7].

4. TOTALLY UMBILICAL CR-SUBMANIFOLDS WITH $q = 1$.

Chen proved the following theorem:

THEOREM 4.1. (Chen [8]). Let M be a totally umbilical CR-submanifold of a Kaehler manifold $\bar{M}$. Then

(i) M is totally geodesic, or

(ii) $q = 1$, or

(iii) M is totally real.

Note that if M were a proper CR-submanifold in the above theorem, then the possibility (iii) would be ruled out. Also note that (i) and (ii) are not mutually exclusive. The case (ii) has been investigated by Chen [8], in the context of a locally Hermitian symmetric space $\bar{M}$ and $\dim \bar{M} \geq 5$. In the following theorem we study the case (ii) by relaxing these conditions and assuming M to be proper.

THEOREM 4.2. Let M be a proper totally umbilical CR-submanifold of a Kaehler manifold $\bar{M}$ with $q = 1$. Suppose the mean curvature vector μ is non-vanishing over M. Then the following statements are equivalent:

(1) M has a normal contact metric (Sasakian) structure [3].

(2) μ has a non-zero constant norm.

(3) μ is parallel in the normal bundle.

(4) second fundamental form of M is parallel.

PROOF. As $\mu \neq 0$ and μ lies in $JD^{\perp}$ by prop. 2.2, it follows that $t\mu \neq 0$ and lies in $D^{\perp}$. Now since $q = 1$, any vector field in $D^{\perp}$ is a scalar multiple of $t\mu$. For any X tangent to M we can show, using (2.7), that

$$g(\mu,\mu)P^2X = g(t\mu,X)t\mu - g(t\mu,t\mu)X$$

Operating P on this gives

$$g(\mu,\mu) = g(t\mu,t\mu) \tag{4.1}$$

Hence we get

$$g(t\mu,t\mu)(P^2X + X) = g(t\mu,X)t\mu \tag{4.2}$$

In this case too, equation (3.1) holds, which shows ($q = 1$) that $g(t\mu,t\mu) \neq 0$ and hence $g(\mu,\mu) \neq 0$. Equation (4.2) becomes

$$P^2X = -X + [g(t\mu,t\mu)]^{-1}g(t\mu,X)t\mu \tag{4.3}$$

As the distribution $D^{\perp}$ is non-degenerate we can, without any loss of generality, assume $g(t\mu,t\mu) = a^2$ where a is a real-valued function on M. Under the setting $\xi = -a^{-2}t\mu$ and η as the dual of ξ, equatin (4.3) takes the form

$$P^2X = -X + \eta(X)\xi \tag{4.4}$$

It follows that $P\xi = 0$, $\eta OP = 0$, rank $(P) = \dim M - 1$ and

$$g(PX,PY) = g(X,Y) - \eta(X)\eta(Y) \tag{4.5}$$

With the help of (2.5), (2.7) and (4.3) we derive

$$(\nabla_X P)Y = a\,\{g(\xi,Y)X - g(X,Y)\xi\} \tag{4.6}$$

$$\nabla_X \xi = a\,PX \tag{4.7}$$

Equations (4.4)-(4.7) show that M has a normal contact metric structure iff. $a^2 = g(\mu,\mu)$ is a non-zero constant. This proves the equivalence of (1) to (2). By virtue of the equality

$$tD_X = 2\ X \ln a\ t\ ,$$

the statement (2) is equivalent to $tD_X\mu = 0$. By differenting the result $f\mu = 0$ (obtained earlier), using (2.8) and operating f^2 on the derived equation we get $f(D_X\mu) = 0$. Thus (2) is equivalent to $D_X\mu = 0$, i.e. the statement (3). The statement (4) means

$$D_X(B(Y,Z)) = B(\nabla_X Y,Z) + B(Y,\nabla_X Z).$$

Substitution $B(X,Y) = g(X,Y)\mu$ in the above shows that (4) is equivalent to (3). This completes the proof.

REMARK 2. Let us compare theorem 4.2 with the following theorem of Chen [8]: "Let M be a totally umbilical CR-submanifold of a locally Hermitian symmetric space $\bar{M}$. If $\dim M \geq 5$ and $q = 1$, then

(a) μ is parallel

(b) if M is not totally geodesic, M is locally isometric to a sphere of radius $1/a$ and rank $\bar{M} \geq \dim M + 1$. Moreover, M is a totally umbilical hypersurface of a

flat, totally geodesic submanifold of $\bar{M}$."

We observe that theorem 4.2 is consistent with the above mentioned theorem. In fact, the conclusion (b) of the theorem shows that M is Sasakian, which is the statement (1) of theorem 4.2. Also the conclusion (a) is the statement (3) of theorem 4.2.

ACKNOWLEDGEMENT. We thank the Natural Sciences and Engineering Research Council of Canada for supporting with a research grant. We are also thankful to the referee for valuable suggestions for the improvement in the original version of this paper.

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