SOME LIMIT THEOREMS FOR RATIOS OF TRANSITION PROBABILITIES

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ABSTRACT. In this paper, infinite dimensional forward convergent stochastic chains have been considered in the framework of [1]. The main result of this paper deals with the observation that the total flow of probability from the C-states to the T-states is very small compared to that from the T-states to the C-states, if the chain is observed for a sufficiently long time. Some examples have been given to justify the assumptions involved.

KEY WORDS AND PHRASES. Infinite stochastic matrices, convergence, basis, C-states, T-States.

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1. INTRODUCTION.

Let (P_n) be a sequence of finite or countably infinite stochastic matrices. Then (P_n) is called a convergent chain iff for each $k \geq 1$, $\lim_{n \rightarrow \infty} P_{k,n} = Q_k$ exists and is a stochastic matrix. Here $P_{k,n} = P_{k+1} P_{k+2} \dots P_n$ for $k < n$. It has been proved in [1] that for such a chain there exists a unique partition $\{T, C_1, C_2, \dots\}$ of the state space S such that for any limit point Q of the Q_k 's the following are true:

$$\begin{aligned} Q_{ij} &= 0, \text{ for all } i \text{ if } j \in T \\ &= 0, \text{ if } i \text{ and } j \text{ belong to different C-classes} \\ &= Q_{kj}, \text{ if } i, j, k \text{ belong to the same C-classes.} \end{aligned}$$

One of the problems, here, is to identify the T-states and the C-states for infinite convergent chains. Also, the following theorem is known (see [2]) for finite chains.

THEOREM 1.1. Let (P_n) be a finite convergent chain with basis $\{T, C_1, C_2, \dots, C_p\}$. Let $L \in T$. Then for each positive integer $k \geq 1$,

$$\lim_{m \rightarrow \infty} \frac{\sum_{n=k}^m (P_n)_{TL}^C}{\sum_{n=k}^m (P_n)_{LL}^C} = 0$$

where $(p_n)_{AB} = \sum_{i \in A} \sum_{j \in B} (p_n)_{ij}$.

The above theorem and some other results have been extended to the infinite-dimensional case with some modifications. Some examples have been discussed to justify the assumptions involved.

2. PRELIMINARIES.

CONDITION (U). A convergent, forward stochastic chain (P_n) is said to satisfy condition (U) if for each j in each C-class in the basis,

$$\lim_{k \rightarrow \infty} \sup_{i \in T} |(Q_k)_{ij} - Q_{ij}| = 0$$

This condition yielded a number of finite space like results for infinite chains in [1]. However, the next few examples demonstrate that Theorem 1.1 fails to go over directly to the infinite state space, even for infinite chains which satisfy condition (U).

EXAMPLES. (a) For a convergent, infinite chain with basis $\{T, C_1, C_2, \dots\}$ and any $L \subseteq T$ the ratio

$$\frac{\sum_{q=k}^m (P_q)_{TL}^C}{\sum_{q=k}^m (P_q)_{LL}^C}$$

for $k \geq 1$, may not go to zero as $m \rightarrow \infty$. For, if

$$P_n = \left[\begin{array}{cccc} 0 & 1/2 & 1/2^2 & \dots \\ 0 & 1/2 & 1/2^2 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 1/2 & 1/2^2 & \dots \\ 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \end{array} \right] \left. \vphantom{\begin{array}{c} \\ \\ \\ \\ \end{array}} \right\} \begin{array}{l} \text{first } n \text{ rows} \end{array}$$

then it has been shown in [1], that for each k ,

$$\lim_{n \rightarrow \infty} P_{k,n} = \left[\begin{array}{ccc} 0 & 1/2 & 1/2^2 \dots \\ 0 & 1/2 & 1/2^2 \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{array} \right]$$

The basis of this chain is of course $\{T = \{1\}, C = \{2, 3, \dots\}\}$. But for all $k < m$,

$$\frac{\sum_{n=k}^m (P_n)_{T\{1\}}^C}{\sum_{n=k}^m (P_n)_{\{1\}T}^C} = \infty$$

It can be seen that this chain satisfies a condition stronger than (U) because $\|Q_k - Q\| = 0$ for all k .

(b) This example is given to show that the previous limit may be finite but not zero.

If $a_k = 2^{4(k-1)}$, $k \geq 1$ with $a_k \leq n < a_{k+1}$, let

$$(P_{3n-2})_{ij} = \begin{cases} 1/2^j & \text{if } i = 0, j \geq 1 \\ 1-1/2^k & \text{if } 1 \leq i = j \leq k \\ 1 & \text{if } i = j \geq k+1 \\ 1/2^k & \text{if } j = n \text{ and } 1 \leq i \leq k \end{cases}$$

$$(P_{3n-1})_{ij} = \begin{cases} 1 & \text{if } 0 \leq i = j \leq k \\ 1 & \text{if } j = n, i > k \end{cases}$$

$$(P_{3n})_{ij} = \begin{cases} 1 & \text{if } i = n, j = 0 \\ 1 & \text{if } i = j \neq n \end{cases}$$

Here, the state space is $\{0, 1, 2, \dots\}$. It has been shown in [1] that

$$\lim_{n \rightarrow \infty} P_{k,n} = \begin{bmatrix} 0 & 1/2 & 1/2^2 & \dots \\ 0 & 1/2 & 1/2^2 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

for all k and $\|Q_k - Q\| = 0$. Here, $T = \{0\}$, $C = \{1, 2, \dots\}$. But

$$\frac{\sum_{n=3k+1}^{3m} (P_n)_{T^c, \{0\}}}{\sum_{n=3k+1}^{3m} (P_n)_{\{0\}, T^c}} = \frac{m-k}{m-k} = 1, \text{ for } k < m$$

so that

$$\lim_{m \rightarrow \infty} \frac{\sum_{n=k}^m (P_n)_{T^c, \{0\}}}{\sum_{n=k}^m (P_n)_{\{0\}, T^c}} = 1, \text{ for all } k$$

so that the limiting value of the above ratio is not zero.

3. MAIN RESULTS.

LEMMA 3.1. Let (P_n) be any chain (finite or infinite). Also, let

$T = \{j \mid \lim_{m \rightarrow \infty} (p_{m,n})_{ij} = 0, \text{ for all } m \text{ and for all } i\}$
 and assume that $Q_k = \lim_{l \rightarrow \infty} P_{k,l}$ are stochastic matrices. Then we have

$$\sum_n (P_n)_{LL}^c = \infty, \text{ where } (P_n)_{LL}^c = \sum_{\substack{i \in L \\ j \in L^c}} (P_n)_{ij} \text{ and } L \neq T.$$

PROOF. Suppose that the lemma is false. Then there exists $L \neq T$ such that $\sum_n (P_n)_{LL}^c < \infty$. Then for given $\epsilon > 0$, there exists $k(\epsilon)$ such that for all $k \geq k(\epsilon)$,

$$\sum_{n \geq k} (P_n)_{LL}^c < \epsilon.$$

For any $t \in L$, $(P_{k,n})_{tL}^c = (P_{k+1} P_{k+1,n})_{tL}^c$

$$\begin{aligned} &= \sum_{i \in S} (P_{k+1})_{ti} (P_{k+1,n})_{iL}^c \\ &= \sum_{i \in L} (P_{k+1})_{ti} (P_{k+1,n})_{iL}^c + \sum_{i \in L^c} (P_{k+1})_{ti} (P_{k+1,n})_{iL}^c \\ &\leq \max_{i \in L} (P_{k+1,n})_{iL}^c + (P_{k+1})_{LL}^c. \end{aligned}$$

Therefore, $\max_{t \in L} (P_{k,n})_{tL}^c \leq \max_{t \in L} (P_{k+1,n})_{tL}^c + (P_{k+1})_{LL}^c$.

Repeating the procedure,

$$\max_{t \in L} (P_{k,n})_{tL}^c \leq \sum_{j=1}^n (P_{k+j})_{LL}^c$$

Therefore, for $k \geq k(\epsilon)$ and $t \in L$,

$$\sum_{j \in L^c} (Q_k)_{tj} < \epsilon$$

which is a contradiction. Hence the supposition is false and the result follows.

REMARK 3.2. If π_{k_0} is an infinite-dimensional initial probability row-vector and (P_n) a sequence of infinite stochastic matrices, then let $\pi_m = \pi_{k_0} P_{k_0,m}$. If π_{k_0} has all positive entries, then for each $j \in T$, choosing k_0 sufficiently large it follows that for a convergent chain, $\lim_{n \rightarrow \infty} \pi_n(j) > 0$. Also, for $m < n$ and $i \in S$,

$$\pi_{n+1}(i) - \pi_n(i) = \sum_{j \neq i} \pi_n(j) (P_{n+1})_{ji} - \pi_n(i) [1 - (P_{n+1})_{ii}] \quad (3.1)$$

$$\pi_n(i) - \pi_m(i) = \sum_{j \neq i} \sum_{k=m}^{n-1} \pi_k(j) (P_{k+1})_{ji} - \sum_{k=m}^{n-1} \pi_k(i) [1 - (P_{k+1})_{ii}] \quad (3.2)$$

The above identities are used to get the following results:

THEOREM 3.3. Let (P_n) be a convergent chain with basis $\{T, C_1, C_2, \dots\}$.

(A) Suppose that $t \in T$. Then the following hold:

$$(i) \quad \sum_{n=1}^{\infty} [1 - (P_n)_{tt}] = \infty$$

$$(ii) \quad \text{For } j \in T, \quad \lim_{n \rightarrow \infty} \frac{(P_n)_{jt}}{1 - (P_n)_{tt}} = 0$$

(B) Suppose that there is a state $t \in S$ such that

$$(i) \quad \sum_{n=1}^{\infty} [1 - (P_n)_{tt}] = \infty$$

$$(ii) \quad \lim_{n \rightarrow \infty} \frac{(P_n)_{jt}}{1 - (P_n)_{tt}} = 0, \quad \text{uniformly for } j \neq t.$$

Then $t \in T$.

(C) Suppose that there is only one T-state such that

$$\lim_{n \rightarrow \infty} \frac{(P_n)_{jt}}{1 - (P_n)_{tt}} = A_j \quad (0 \leq A_j < \infty) \quad \text{for each } j \neq t.$$

Then $t \in T$ iff $\sum_{n=1}^{\infty} [1 - (P_n)_{tt}] = \infty$ and $A_j = 0$ for each $j \neq t$.

PROOF. (A) Part (i) follows since $(P_{k,n})_{tt} \geq (P_{k+1})_{tt} (P_{k+2})_{tt} \dots (P_n)_{tt}$

So that $0 = (Q_k)_{tt} \geq \prod_{m=k+1}^{\infty} (P_{k+m})_{tt}$

which means that $\sum_{n=1}^{\infty} [1 - (P_n)_{tt}] = \infty$.

For part (ii), suppose that it is false for some $j_0 \in T$. Then there exists $\beta > 0$ and a positive integer N such that for all $n \geq N$,

$$(P_n)_{j_0 t} \geq \beta [1 - (P_n)_{tt}].$$

Then from equation (3.2) for $n > m > N$, we have

$$\pi_n(t) - \pi_m(t) \geq \sum_{k=m}^{n-1} [\beta \pi_k(j_0) - \pi_k(t)] [1 - (P_{k+1})_{tt}]$$

which is impossible because of part (i) and because the left side goes to zero as $m, n \rightarrow \infty$ but $\lim_{k \rightarrow \infty} \pi_k(j_0) > 0$ and $\lim_{k \rightarrow \infty} \pi_k(t) = 0$. Therefore the supposition is false and part (ii) is proved

(B) Suppose $t \in T$. Then there exists n_0 such that $n \geq n_0$ implies

$$\pi_n(t) > 2\beta > 0.$$

By (1), $2\beta [1 - (P_{n+1})_{tt}]$

$$\leq \sum_{j \neq t} \pi_n(j)(P_{n+1})_{jt} + \pi_n(t) - \pi_{n+1}(t)$$

$$\leq \max_{j \neq t} (P_{n+1})_{jt} + \pi_n(t) - \pi_{n+1}(t).$$

By condition (ii), there exists $N > n_0$ such that $n \geq N$ implies

$$\max_{j \neq t} (P_{n+1})_{jt} < \beta [1 - (P_n)_{tt}]$$

then we have from above, for $n > N$,

$$\beta [1 - (P_{n+1})_{tt}] \leq \pi_n(t) - \pi_{n+1}(t)$$

which means that $\beta \sum_{n=N+1}^m [1 - (P_{n+1})_{tt}] \leq \pi_{N+1}(t) - \pi_{m+1}(t) < 2$
which contradicts (i).

(C) This part follows immediately from parts (A) and (B).

THEOREM 3.4. Let (P_n) be a convergent infinite chain with $\{T, C_1, C_2, \dots\}$ as its basis. Then for all $k \geq 1$,

$$\lim_{m \rightarrow \infty} \frac{\sum_{n=k}^m (P_n)_{BA}}{\sum_{n=k}^m (P_n)_{AA^c}} = 0$$

where A (finite set) T, B (finite set) T^c . Moreover, if

$$\lim_{m \rightarrow \infty} [\inf_{t \in T} \sum_{i \notin C_s} (Q_n)_{ti}] > 0$$

then the above result holds for any $A \subset T$ and finite $B \subset C_s$.

PROOF. If $\pi_n(A) = \sum_{t \in A} \pi_n(t)$ and $m < n$, then

$$\pi_n(A) - \pi_m(A) = \sum_{t \in A} \sum_{j \neq t} \sum_{q=m}^{n-1} \pi_q(j)(P_{q+1})_{jt}$$

$$\begin{aligned}
& - \sum_{t \in A} \sum_{q=m}^{n-1} \pi_q(j) \sum_{j \neq t} (P_{q+1})_{tj} \\
& \geq \sum_{q=m}^{n-1} \sum_{j \in B} \pi_q(j) (P_{q+1})_{jA} - \sum_{q=m}^{n-1} \sum_{t \in A} \pi_q(t) (P_{q+1})_{tA}^c
\end{aligned}$$

Now, for all $t \in A$ (finite) and $j \in B$ (finite) and q sufficiently large,

$$\pi_q(t) < \epsilon \epsilon_1 \quad \text{and} \quad \pi_q(j) > \epsilon_1.$$

Therefore, for m sufficiently large and $m < n$,

$$\pi_n(A) - \pi_m(A) > \epsilon_1 \sum_{q=m}^{n-1} (P_{q+1})_{BA} - \epsilon \epsilon_1 \sum_{q=m}^{n-1} (P_{q+1})_{AA}^c \quad (3.3)$$

Let (m_r) be any given sequence of positive integers. Then there exists (p_r) (m_r) such that for each $r \geq k$, there exists $\beta > 0$ such that

$$\sum_{q=k}^r (P_q)_{BA} \leq \epsilon/2 \sum_{q=k}^{p_r} (P_q)_{AA}^c \quad (3.4)$$

$$\text{since} \quad \sum_q (P_q)_{AA}^c = \infty \quad \text{and} \quad \sum_{q=r}^{p_r} (P_q)_{AA}^c > \beta > 0. \quad (3.5)$$

$$\text{But} \quad \lim_{r \rightarrow \infty} \frac{\sum_{q=r}^{p_r} (P_q)_{BA}}{\sum_{q=r}^{p_r} (P_q)_{AA}^c} = 0 \quad \text{from equation (3.3). Hence from (3.4) and (3.5),}$$

for sufficiently large r , we have

$$\sum_{q=k}^{p_r} (P_q)_{BA} < \epsilon \sum_{q=k}^{p_r} (P_q)_{AA}^c$$

and the first part of the theorem follows.

We refer the reader to a result obtained in [2] for the proof of the second part. It has been proved in [2] that for all $i \in C_s$,

$$\sum_{n=1}^{\infty} \sum_{t \in A} (P_n)_{it} < \infty.$$

Using the above result, the proof of the second part easily follows.

EXAMPLE 3.5. We now give an example to show that in Theorem 3.4, the condition that the flow in the denominator is from A to A^c , is necessary.

$$\text{Let } P_n = \begin{bmatrix} 0 & 1/n & 1-1/n & 0 & \dots\dots \\ 1/n & 1-1/n & 0 & 0 & \dots\dots \\ 0 & 1 & 0 & 0 & \dots\dots \\ 0 & 1 & 0 & 0 & \dots\dots \end{bmatrix}$$

By Bernstein's theorem, the products are weakly ergodic. In fact,

$$\begin{aligned} (P_{k,n+1})_{i1} &= \sum_s (P_{k,n})_{is} (P_{n+1})_{s1} \\ &= \frac{1}{n+1} (P_{k,n})_{i2} \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for all } i \\ (P_{k,n+1})_{i2} &= \sum_s (P_{k,n})_{is} (P_{n+1})_{s2} \\ &= 1 - (P_{k,n})_{i1} + \frac{1}{n+1} (P_{k,n})_{i1} - \frac{1}{n+1} (P_{k,n})_{i2} \\ &\rightarrow 1 \text{ as } n \rightarrow \infty. \end{aligned}$$

So, the chain is strongly ergodic with $T^c = \{2\}$. But if $A = \{1\}$, $B = \{2\}$

$$\text{then } \frac{\sum_{n=k}^m (P_n)_{21}}{\sum_{n=k}^m (P_n)_{12}} = 1, \text{ for all } k \leq m.$$

Hence, the assertion is true.

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