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DIFFERENTIABLE STRUCTURES ON A GENERALIZED PRODUCT OF SPHERES

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ABSTRACT. In this paper, we give a complete classification of smooth structures on a generalized product of spheres. The result generalizes our result in [1] and R. de Sapio's result in [2].

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1. INTRODUCTION

In [2] a classification of smooth structures on product of spheres of the form $S^k \times S^p$ where $2 \leq k \leq p$, $k+p \geq 6$ was given by R. de Sapio and in [1] this author extended R. de Sapio's result to smooth structures on $S^p \times S^q \times S^r$ where $2 \leq p \leq q \leq r$. The next question is, how many differentiable structures are there in any arbitrary product of ordinary spheres. In this paper, we give a classification under the relation of orientation preserving diffeomorphism of all differentiable structures of spheres $S^{k_1} \times S^{k_2} \times \ldots \times S^{k_r}$ where $2 \leq k_1 < k_2 \leq \ldots \leq k_r$. S^n denotes the unit n-sphere with the usual differential structure in the Euclidean (n+1)-space R^{n+1}. θ^n denotes the group of h-cobordism classes of homotopy n-sphere under the connected sum operation. Σ^n will denote an homotopy n-sphere. $H(p,k)$ denotes the subset of θ^p which consists of those homotopy p-sphere Σ^p such that $\Sigma^p \times S^k$ is diffeomorphic to $S^p \times S^k$. By [2], $H(p,k)$ is a subgroup of θ^p and it is not always zero and in fact in [1], we showed that if $k \geq p-3$, then $H(p,k) = \theta^p$.

By Hauptremutung [3], piecewise linear homoemorphism will be replaced by homeomorphism. Consider two manifolds $S^{k_1} \times S^{k_2} \times S^{k_3} \times S^{k_4}$ and $\Sigma^{k_2+k_4} \times S^{k_1} \times S^{k_3}$, we shall denote the connected sum of the two manifolds along a k_2+k_4-1 cycle by

$(S^{k_1}\times S^{k_2}\times S^{k_3}\times S^{k_4})\#_{k_2+k_4}(\Sigma^{k_2+k_4}\times S^{k_1}\times S^{k_3})$. This is geometrically achieved by removing $\mathrm{Int}(D^{k_2+k_4})\times S^{k_1}\times S^{k_3}$ from both manifolds and then identify their common boundary. Thus nothing else other than taking the usual connected sum of $S^{k_2}\times S^{k_4}$ and $\Sigma^{k_2+k_4}$ by removing the interior of an embedded disc $D^{k_2+k_4}$ from each manifold and identify the manifolds along their common boundary $S^{k_2+k_4-1}$ to obtain $S^{k_2}\times S^{k_4}\#\Sigma^{k_2+k_4}$. This is a well-defined operation. We then take the cartesian product with $S^{k_1}\times S^{k_3}$ to have $S^{k_1}\times S^{k_3}\times(S^{k_2}\times S^{k_4}\#\Sigma^{k_2+k_4})=$ $S^{k_1}\times S^{k_3}\times S^{k_2}\times S^{k_4}\#_{k_2+k_4}\Sigma^{k_2+k_4}\times S^{k_1}\times S^{k_3}$. But $S^{k_1}\times S^{k_3}\times S^{k_2}\times S^{k_4}$ is diffeomorphic to $S^{k_1}\times S^{k_2}\times S^{k_3}\times S^{k_4}$ then $(S^{k_1}\times S^{k_2}\times S^{k_3}\times S^{k_4})\#_{k_2+k_4}\Sigma^{k_2+k_4}\times S^{k_1}\times S^{k_3}=S^1\times S^{k_3}\times(S^{k_2}\times S^{k_4}\#\Sigma^{k_2+k_4})$.

We will then prove the following.

CLASSIFICATION THEOREM If M^n is a smooth manifold homeomorphic to $S^{k_1}\times S^{k_2}\times\ldots\times S^{k_r}$ where $2\le k_1<\ldots<K_{r-1}$ and $k_4-3\le k_{r-1}\le k_r$ and $n=k_1+k_2+\ldots+k_r$ then there exists homotopy spheres $\Sigma^{k_1+k_2+k_3},\ldots\Sigma^{n-k_r},\ldots,\Sigma^{n-k_1},\Sigma^n$ such that M^n is diffeomorphic to

$$\Big[(S^{k_1}\times\ldots\times S^{k_r})\#_{k_1+k_r}(\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}})\#_{k_2+k_r}(\Sigma^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_{r-1}})$$
$$\#\ldots\#_{k_1+k_2+k_3}(\Sigma^{k_1+k_2+k_3}\times S^{k_4}\times\ldots\times S^{k_r})\#\ldots\#_{n-k_r}(\Sigma^{n-k_r}\times S^{k_r})$$
$$\#\ldots\#_{n-k_1}(\Sigma^{n-k_1}\times S^{k_1})\Big]\#\Sigma^n .$$

We shall use the above classification theorem to give the number of differentiable structures on $S^{k_1}\times S^{k_2}\times\ldots\times S^{k_r}$. We shall lastly compute the number of structures in some simple cases.

2. PRELIMINARY RESULTS

We shall apply obstruction theory of Munkres [4]. Let M and N be smooth n-manifolds and L a closed subset of M when triangulated. A homeomorphism $f:M\to N$ is a diffeomorphism modulo L if $f|(M-L)$ is a diffeomorphism and each simplex α of L has a neighborhood V, such that f is smooth on $V-L$ near α. By [4], if two n-manifolds M and N are combinatorially equivalent then M is diffeomorphic modulo an (n-1)-skeleton L onto N.

If $f:M^n\to N^n$ is a diffeomorphism modulo m-skeleton $m<n$ then Munkres showed that the obstruction to deforming f to a diffeomorphism $g:M^n\to N^n$ modulo (m-1)-skeleton is an element $\lambda_m(f)\in H_m(M,\Gamma^{n-m})=\Gamma^{n-m}$. Where Γ^{n-m} is a group of diffeomorphisms of S^{n-m-1} modulo the diffeomorphisms that are extendable to diffeomorphisms of D^{n-m}. We call g the smoothing of f. If $\lambda_m(f)=0$ then g exists. Recall that in ([1], Lemma 2.1.1) we proved that if $q\ge p$ then $\Sigma^p\times S^q$ is diffeomorphic to $S^p\times S^q$ for any homotopy sphere Σ^p. In Remark (1) following that lemma, we showed further that even when $p-3\le q$ the result is still true.

LEMMA 2.1 Suppose $f:M^n\to S^{k_1}\times S^{k_2}\times\ldots\times S^{k_r}$ is a piecewise linear homeomorphism which is a diffeomorphism modulo $(n-k_i)$-skeleton $1\le i\le r$, then there exists an

homotopy sphere Σ^{k_i} and a piecewise linear homeomorphism

$$h : M^n \to S^{k_1} \times S^{k_2} \times \dots \times S^{k_{i-1}} \times \Sigma^{k_i} \times S^{k_{i+1}} \times \dots \times S^{k_r}$$

which is a diffeomorphism modulo $(n-k_i-1)$ skeleton.

PROOF. Since $f : M^n \to S^{k_1} \times \dots \times S^{k_r}$ is a diffeomorphism modulo $(n-k_i)$-skeleton then by Munkres [4], the obstruction to deforming f to a diffeomorphism modulo $(n-k_i-1)$-skeleton is an element $\lambda_{k_i}(f) \in H_{n-k_i}(M^n, \Gamma^{k_i}) = \Gamma^{k_i}$. Let $[\psi] = \lambda_{k_i}(f) \in \Gamma^{k_i}$ where $\psi : S^{k_i-1} \to S^{k_i-1}$ is a diffeomorphism. We define $\Sigma^{k_i} = D_1^{k_i} \cup_\psi D_2^{k_i}$ and a homeomorphism $j : S^{k_i} \to \Sigma^{k_i}$ where we have $S^{k_i} = D_1^{k_i} \cup_{id} D_2^{k_i}$ and so j is identity map on $\mathrm{Int}(D_1^{k_i})$ and radial extension of ψ^{-1} on $D_2^{k_i}$. So j is a piecewise linear homeomorphism by the definition and the obstruction to deforming j to a diffeomorphism is $[\psi^{-1}] = -\lambda_{k_i}(f)$. So consider the map

$$id \times j : (S^{k_1} \times \dots \times S^{k_i-1} \times S^{k_i+1} \times \dots \times S^{k_r}) \times S^{k_i} \to (S^{k_1} \times \dots \times S^{k_i-1} \times S^{k_i+1} \times \dots \times S^{k_r}) \times \Sigma^{k_i} .$$

The map is a piecewise linear homeomorphism and the obstruction to deforming it to a diffeomorphism is $[\psi^{-1}] = -\lambda_{k_i}(f)$. Notice that the manifold $(S^{k_1} \times \dots \times S^{k_i-1} \times \dots \times S^{k_r}) \times S^{k_i} = S^{k_1} \times \dots \times S^{k_{i-1}} \times S^{k_i} \times S^{k_{i+1}} \times \dots \times S^{k_r}$ and $(S^{k_1} \times \dots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \dots \times S^{k_r}) \times \Sigma^{k_i} = S^{k_1} \times \dots \times S^{k_{i-1}} \times \Sigma^{k_i} \times S^{k_{i+1}} \times \dots \times S^{k_r}$.

Consider the composite $(id \times j) \cdot f = h$, the obstruction to deforming h to a diffeomorphism modulo $(n-k_i-1)$ skeleton is $\lambda_{k_i}(h) = \lambda_{k_i}((id \times j) \cdot f) = \lambda_{k_i}(id \times j) + \lambda_{k_i}(f) = -\lambda_{k_i}(f) + \lambda_{k_i}(f) = 0$ hence $h : M^n \to S^{k_1} \times \dots \times S^{k_{i-1}} \times \Sigma^{k_i} \times S^{k_{i+1}} \times \dots \times S^{k_r}$ is a diffeomorphism modulo $(n-k_i-1)$ skeleton. Hence the lemma.

LEMMA 2.2 Let $f : M^n \to S^{k_1} \times \dots \times S^{k_r}$ be a diffeomorphism modulo $n-(k_i+k_j)$ skeleton $1 \le i, j \le r$ then there exists homotopy sphere $\Sigma^{k_i+k_j}$ and a piecewise linear homeomorphism

$$f : M^n \to (S^{k_1} \times \dots \times S^{k_r}) \#_{k_i+k_j} (\Sigma^{k_i+k_j} \times S^{k_1} \times \dots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \dots \times S^{k_{j-1}} \times S^{k_{j+1}} \times \dots \times S^{k_r})$$

which is a diffeomorphism modulo $n-(k_i+k_j)-1$ skeleton.

PROOF. Since f is a diffeomorphism modulo $n-(k_i+k_j)$ skeleton, it follows that the obstruction to deforming f to a diffeomorphism modulo $n-(k_i+k_j)-1$ skeleton is

$$\lambda(f) \in H_{n-(k_i+k_j)}(M^n, \Gamma^{k_i+k_j}) = \Gamma^{k_i+k_j} . \quad \text{Let } [\phi] = \lambda(f) \in \Gamma^{k_i+k_j}$$

where $\phi : S^{k_i+k_j-1} \to S^{k_i+k_j-1}$ is a diffeomorphism and $\Sigma^{k_i+k_j} = D^{k_i+k_j} \cup_\phi D^{k_i+k_j}$. We define $j : S^{k_i} \times S^{k_j} \to S^{k_i} \times S^{k_j} \# \Sigma^{k_i+k_j}$ to be identity map on $S^{k_i} \times S^{k_j} - \mathrm{Int}(D^{k_i+k_j})$ and radial extension of ϕ^{-1} on $\mathrm{Int}(D^{k_i+k_j})$ hence j is a piecewise linear homeomorphism and the obstruction to deforming j to a diffeomorphism is $[\phi^{-1}] = -\lambda(f)$. Then consider

$$j \times id : S^{k_i} \times S^{k_j} \times (S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_{j-1}} \times S^{k_{j+1}} \times \ldots \times S^{k_r}) \longrightarrow$$
$$(S^{k_i} \times S^{k_j} \#_{\Sigma^{k_i+k_j}}) \times (S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_{j-1}} \times S^{k_{j+1}} \times \ldots \times S^{k_r}) .$$

Note that

$$S^{k_i} \times S^{k_j} \times (S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_{j-1}} \times S^{k_{j+1}} \times \ldots \times S^{k_r}) =$$
$$(S^{k_1} \times S^{k_2} \times \ldots \times S^{k_i} \times \ldots \times S^{k_j} \times \ldots \times S^{k_r})$$

and

$$(S^{k_i} \times S^{k_j} \#_{\Sigma^{k_i+k_j}}) \times (S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_r}) =$$
$$(S^{k_1} \times \ldots \times S^{k_r}) \#_{k_i+k_j} (\Sigma^{k_i+k_j} \times S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_{j-1}} \times S^{k_{j+1}} \times \ldots \times S^{k_r})$$

hence the above map is

$$id \times j : (S^{k_1} \times \ldots \times S^{k_r}) \longrightarrow$$
$$(S^{k_1} \times \ldots \times S^{k_r}) \#_{k_i+k_j} (\Sigma^{k_i+k_j} \times S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_{j-1}} \times S^{k_{j+1}} \times \ldots \times S^{k_r})$$

which is piecewise linear and its obstruction to a diffeomorphism is $-\lambda(f)$ hence the obstruction to deforming the composite $(j \times id)\cdot f$ to a diffeomorphism modulo $n-(k_i+k_j)-1$ skeleton is zero. Hence if $f' = (j \times id)\cdot f$ then $f' : M^n \longrightarrow (S^{k_1} \times \ldots \times S^{k_r}) \#_{k_i+k_j} (\Sigma^{k_i+k_j} \times S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_{j-1}} \times S^{k_{j+1}} \times \ldots \times S^{k_r})$ is a diffeomodphism modulo $n-(k_i+k_j)-1$ skeleton.

3. CLASSIFICATION

THEOREM 3.1 If M^n is a smooth manifold homeomorphic to $S^{k_1} \times S^{k_2} \times \ldots \times S^{k_r}$ then there exists homotopy spheres, $\Sigma^{k_1+k_r}, \Sigma^{k_2+k_r}, \ldots, \Sigma^{k_1+k_2+k_3}, \ldots, \ldots \Sigma^{n-k_r}, \ldots \Sigma^{n-k_1}$, and Σ^n such that M^n is diffeomorphic to

$$(S^{k_1} \times \ldots \times S^{k_r}) \#_{k_1+k_r} (\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \#_{k_2+k_r} (\Sigma^{k_2+k_r} \times S^{k_1} \times S^{k_3} \times \ldots \times S^{k_{r-1}})$$
$$\# \ldots \#_{k_1+k_2+k_3} (\Sigma^{k_1+k_2+k_3} \times S^{k_4} \times \ldots \times S^{k_r})$$
$$\ldots \#_{n-k_r} (\Sigma^{n-k_r} \times S^{k_r}) \# \ldots \#_{n-k_1} (\Sigma^{n-k_1} \times S^{k_1}) \# \Sigma^n$$

where $2 \le k_1 < k_2 < \ldots < k_r$, $k_r - 3 \le k_{r-1} \le k_r$ and $n = k_1 + k_2 + \ldots + k_r$.

PROOF. Suppose $M^n \xrightarrow{h} S^{k_1} \times \ldots \times S^{k_r}$ is the homeomorphism. By Munkres theory [4], h is a diffeomorphism modulo $(n-1)$ skeleton. Since the first non-zero homology appears in dimension $n-k_1$, (apart from the zero dimension) it then means that h is a diffeomorphism modulo $(n-k_1)$ skeleton. The obstruction to deforming h to a diffeomorphism modulo $(n-k_1-1)$ skeleton is $\lambda(h) \in H_{n-k_1}(M^n, \Gamma^{k_1}) = \Gamma^{k_1}$. By Lemma 2.1, there exists a piecewise linear homeomorphism h' and a homotopy sphere Σ^{k_1} such that $h' : M^n \to \Sigma^{k_1} \times S^{k_2} \times \ldots \times S^{k_r}$ which is a diffeomorphism modulo $(n-k_1-1)$

skeleton. In [1] Lemma 2.1.1 it was proved that $\Sigma^{k_1} \times S^{k_2}$ is diffeomorphic to $S^{k_1} \times S^{k_2}$ since $k_1 < k_2$. It then follows that $\Sigma^{k_1} \times S^{k_2} \times \ldots \times S^{k_r}$ is diffeomorphic to $S^{k_1} \times \ldots \times S^{k_r}$ hence $h' : M^n \to S^{k_1} \times \ldots \times S^{k_r}$ is a diffeomorphism modulo $(n-k_1-1)$ skeleton. There is no other obstruction to deforming h' to a diffeomorphism until the $(n-k_2-1)$-skeleton. This is because

$$H_i(M^n, Z) = 0$$

for $n-k_2+1 < i < n-k_1$. So we can assume that h' is a diffeomorphism modulo $(n-k_2)$ skeleton. The obstruction to deforming h' to a diffeomorphism modulo $(n-k_2-1)$ skeleton is $\lambda(h') \in H_{n-k_2}(M^n, \Gamma^{k_2}) = \Gamma^{k_2}$. Again by Lemma 2.1, there exists a homotopy sphere Σ^{k_2} and a piecewise linear homeomorphism $h'' : M^n \to S^{k_1} \times \Sigma^{k_2} \times S^{k_3} \times \ldots \times S^{k_r}$ which is a diffeomorphism modulo $(n-k_2-1)$ skeleton. By the same argument as above since $k_2 < k_3$ we see that $\Sigma^{k_2} \times S^{k_3}$ is diffeomorphic to $S^{k_2} \times S^{k_3}$ hence $S^{k_1} \times \Sigma^{k_2} \times S^{k_3} \times \ldots \times S^{k_r}$ is diffeomorphic to $S^{k_1} \times S^{k_2} \times S^{k_3} \times \ldots \times S^{k_r}$. This shows that $h'' : M^n \to S^{k_1} \times \ldots \times S^{k_r}$ is a diffeomorphism modulo $(n-k_2-1)$-skeleton. By the same argument since M^n has no homology between $n-k_3-1$ and $n-k_2-1$ we can assume that h'' is a diffeomorphism modulo $(n-k_3)$-skeleton. Proceeding this way using the same argument we can construct a homeomorphism say $h'' : M^n \to S^{k_1} \times \ldots \times S^{k_r}$ which is a diffeomorphism modulo $(n-k_r)$-skeleton. However, to deform h'' to a diffeomorphism modulo $(n-k_r-1)$-skeleton, there is an obstruction $\lambda(h'') \in H_{n-k_r}(M^n, \Gamma^{k_r}) = \Gamma^{k_r}$. It also follows by Lemma 2.1 that there exists a homotopy sphere Σ^{k_r} and a piecewise linear homeomorphism $f : M^n \to S^{k_1} \times \ldots \times S^{k_{r-1}} \times \Sigma^{k_r}$ which is a diffeomorphism modulo $(n-k_r-1)$-skeleton. Now in Remark (1) of [1] it was shown that even when $p-3 \leq r$, $S^r \times \Sigma^p$ is diffeomorphic to $S^r \times S^p$ and so by our assumption that $k_r-3 \leq k_{r-1} \leq k_r$ it follows that $S^{k_{r-1}} \times \Sigma^{k_r}$ is diffeomorphic to $S^{k_{r-1}} \times S^{k_r}$. Hence $S^{k_1} \times \ldots \times S^{k_{r-1}} \times \Sigma^{k_r}$ is diffeomorphic to $S^{k_1} \times \ldots \times S^{k_{r-1}} \times S^{k_r}$ and so $f : M^n \to S^{k_1} \times \ldots \times S^{k_r}$ is a diffeomorphism modulo $(n-k_r-1)$-skeleton. The next obstruction to deforming f to a diffeomorphism is on $n-(k_r+k_1)-1$ skeleton and it is $\lambda(f) \in H_{n-(k_1+k_r)}(M^n, \Gamma^{k_1+k_r}) = \Gamma^{k_1+k_r}$. By Lemma 2.2, there exists a piecewise linear homeomorphism

$$f' : M^n \to (S^{k_1} \times S^{k_2} \times \ldots \times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}})$$

which is a diffeomorphism modulo $n-(k_1+k_r)-1$ skeleton for some homotopy sphere $\Sigma^{k_1+k_r}$ defined using $\lambda(f) \in \Gamma^{k_1+k_r}$. At this point, we want to remark that if $k_1+k_r-3 \leq \max(k_2, \ldots, k_{r-1})$ and suppose $k_j = \max(k_2, \ldots, k_{r-1})$ then it follows from Remark (1) of [1] since $k_1+k_r-3 \leq k_j$, that $\Sigma^{k_1+k_r} \times S^{k_j}$ is diffeomorphic to $S^{k_1+k_r} \times S^{k_j}$ and so $\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}$ is diffeomorphic to $S^{k_1+k_r} \times S^{k_1} \times \ldots \times S^{k_{r-1}}$. This then implies that $(S^{k_1} \times \ldots \times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}})$ is diffeomorphic to $(S^{k_1} \times S^{k_2} \times \ldots \times S^{k_r}) \underset{k_1+k_r}{\#} (S^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}})$ and this is diffeomorphic to $S^{k_1} \times S^{k_2} \times \ldots \times S^{k_r}$ because $S^{k_1} \times S^{k_r} \# S^{k_1+k_r} = S^{k_1} \times S^{k_r}$. So this means that the factor

$\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}$ will disappear in the above sum if we have the condition $k_1+k_r-3 \leq \max(k_2,\ldots,k_{r-1})$.

Anyway, we have $f' : M^n \to (S^{k_1}\times\ldots\times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}})$ which is a diffeomorphism modulo $n-(k_1+k_r)-1$ skeleton. Since $H_i(M^n,Z) = 0$ for $n-(k_2+k_r) < i \leq n-(k_1+k_r)-1$ then there is no obstruction to deforming f' to a diffeomorphism modulo $n-(k_2+k_r)$ skeleton and the obstruction to deforming f' to a diffeomorphism modulo $n-(k_2+k_r)-1$ skeleton is $\lambda(f') \in H_{n-(k_2+k_r)}(M^n,\Gamma^{k_2+k_r}) = \Gamma^{k_2+k_r}$. Using the same technique as in the proof of Lemma 2.2 it can be easily shown that there exists an homotopy sphere $\Sigma^{k_2+k_r} = D^{k_2+k_r} \underset{\psi}{\cup} D^{k_2+k_r}$ where $\psi = \lambda(f') \in \Gamma^{k_2+k_r}$ and $\psi : S^{k_2+k_r-1} \to S^{k_2+k_r-1}$ is a diffeomorphism and a piecewise linear homeomorphism

$$j : S^{k_1} \times \ldots \times S^{k_r} \to (S^{k_1}\times\ldots\times S^{k_r}) \underset{k_2+k_r}{\#} (\Sigma^{k_2+k_r}\times S^{k_1}\times\ldots\times S^{k_{r-1}})$$

where obstruction to a diffeomorphism is $-\lambda(f')$. We now define a map

$$j' : (S^{k_1}\times\ldots\times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}})$$

$$\to (S^{k_1}\times\ldots\times S^{k_r}) \underset{k_2+k_r}{\#} (\Sigma^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_{r-1}}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}})$$

where $j' = j$ on $(S^{k_1}\times\ldots\times S^{k_r}) - \mathrm{Ind}(D^{k_1+k_r}) \times S^{k_2}\times\ldots\times S^{k_{r-1}}$ and identity on $\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}} - \mathrm{Int}(D^{k_1+k_r}) \times S^{k_2} \times \ldots \times S^{k_{r-1}}$.

Clearly j' is piecewise linear and its obstruction to a diffeomorphism is $-\lambda(f')$ hence the obstruction to deforming the composite $g = j' \cdot f'$ where $g : M^n \to (S^{k_1}\times\ldots\times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}}) \underset{k_2+k_r}{\#} (\Sigma^{k_2+k_r}\times S^{k_1}\times\ldots\times S^{k_{r-1}})$ is $\lambda(j'\cdot f') = \lambda(j') + \lambda(f') = 0$. Hence $g = j' \cdot f'$ is a diffeomorphism modulo $n-(k_2+k_r)-1$ skeleton. Proceeding in this way, we see that the next obstruction to a diffeomorphism will be on $(n-(k_3+k_r))$-skeleton. Using the above technique continuously, we can construct a piecewise linear homeomorphism

$$g' : M^n \to (S^{k_1}\times\ldots\times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r} \times S^{k_2}\times\ldots\times S^{k_{r-1}})$$

$$\underset{k_2+k_r}{\#} (\Sigma^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_{r-1}}) \underset{k_{i_1}+\ldots+k_{i_\ell}}{\#\ldots\#} (\Sigma^{k_{i_1}+\ldots+k_{i_2}} \times S^{k_{j_1}}\times\ldots\times S^{k_{j_p}}) ,$$

$$j_p's \neq i_\ell$$

which is a diffeomorphism modulo $n-(k_{r-1}+\ldots+k_1) = k_r$ skeleton. The obstruction to extending g' to a diffeomorphism modulo (k_r-1) skeleton is $\lambda(g') \in H_{k_r}(M^n,\Gamma^{n-k_r}) = \Gamma^{n-k_r}$. By using the same technique as in the proof of Lemma 2.1, there exists a piecewise linear homeomorphism j and homotopy sphere Σ^{n-k_r} such that

$$j : S^{k_1}\times\ldots\times S^{k_r} \to (S^{k_1}\times\ldots\times S^{k_r}) \underset{n-k_r}{\#} (\Sigma^{n-k_r} \times S^{k_r})$$

has an obstruction to a diffeomorphism to be $-\lambda(g')$. From this we define the map,

$$j':(S^{k_1}\times\ldots\times S^{k_r})\underset{k_1+k_r}{\#}(\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}})$$
$$\underset{k_2+k_r}{\#}(\Sigma^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_{r-1}})\#\ldots\underset{\substack{k_{i_1}+\ldots+k_{i_\ell}\\ j_p\neq i_1,i_2\ldots i_\ell}}{\#}(\Sigma^{k_{i_1}+\ldots+k_{i_\ell}}\times S^{k_{j_1}}\times\ldots\times S^{k_{j_p}})$$
$$\longrightarrow(S^{k_1}\times\ldots\times S^{k_r})\underset{n-k_r}{\#}(\Sigma^{n-k_r}\times S^{k_r})\underset{k_1+k_r}{\#}(\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}})$$
$$\#\ldots\underset{k_{i_1}+\ldots+k_{i_\ell}}{\#}(\Sigma^{k_{i_1}+\ldots+k_{i_\ell}}\times S^{k_{j_1}}\times\ldots\times S^{k_{j_p}})$$

where $j'=j$ on $(S^{k_1}\times\ldots\times S^{k_r})-(\mathrm{Int}(D^{k_1+k_r})\times S^{k_2}\times\ldots\times S^{k_{r-1}})$ and identity elsewhere. It is easily seen that j' is piecewise linear homeomorphism and the obstruction to deforming the composite $j'\cdot g'$ to a diffeomorphism is zero. Hence the map $h'=j'\cdot g'$ where

$$h':M^n\to(S^{k_1}\times\ldots\times S^{k_r})\underset{k_1+k_r}{\#}(\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}})$$
$$\underset{k_2+k_r}{\#}(\Sigma^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_{r-1}})\#\ldots\underset{n-k_r}{\#}(\Sigma^{n-k_r}\times S^{k_r})$$

is a diffeomorphism modulo (k_r-1) skeleton. However, since $H_i(M^n,Z)=0$ for $k_{r-1}<i<k_r-1$, there is no more obstruction to deforming h' to a diffeomorphism modulo k_{r-1}-skeleton. To deform h' to a diffeomorphism modulo $(k_{r-1}-1)$ skeleton, there is an obstruction and this equals $\lambda(h')\in H_{k_{r-1}}(M^n,\Gamma^{n-k_{r-1}})=\Gamma^{n-k_{r-1}}$. Applying the above technique again, we can get an homotopy sphere $\Sigma^{n-k_{r-1}}$ and a piecewise linear homeomorphism

$$h'':M^n\to(S^{k_1}\times\ldots\times S^{k_r})\underset{k_1+k_r}{\#}(\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}})$$
$$\underset{k_2+k_r}{\#}(\Sigma^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_{r-1}})\#\ldots\underset{k_1+k_2+k_3}{\#}(\Sigma^{k_1+k_2+k_3}\times S^{k_4}\times\ldots\times S^{k_r})$$
$$\ldots\underset{n-k_r}{\#}(\Sigma^{n-k_r}\times S^{k_r})\underset{n-k_{r-1}}{\#}(\Sigma^{n-k_{r-1}}\times S^{k_{r-1}})$$

which is a diffeomorphism modulo $k_{r-1}-1$ skeleton. The next obstruction will be on $k_{r-2}-1$ skeleton. Proceeding this way gradually down the remaining skeleton, we can construct a map

$$g:M^n\to(S^{k_1}\times\ldots\times S^{k_r})\underset{k_1+k_r}{\#}(\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}})$$
$$\underset{k_2+k_r}{\#}(\Sigma^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_{r-1}})\#\ldots\underset{k_1+k_2+k_3}{\#}(\Sigma^{k_1+k_2+k_3}\times S^{k_4}\times\ldots\times S^{k_r})$$
$$\ldots\underset{n-k_r}{\#}(\Sigma^{n-k_r}\times S^{k_r})\underset{n-k_{r-1}}{\#}(\Sigma^{n-k_{r-1}}\times S^{k_{r-1}})\#\ldots\underset{n-k_1}{\#}(\Sigma^{n-k_1}\times S^{k_1})$$

which is a diffeomorphism modulo k_1-skeleton. Since $H_i(M^n,Z)=0$ for $0<i<k$, then g is a diffeomorphism modulo one point. It therefore follows that there exist an homotopy sphere Σ^n such that M^n is diffeomorphis to

$$\Big[(S^{k_1}\times\ldots\times S^{k_r} \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_r-1}) \underset{k_2+k_r}{\#} (\Sigma^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_r-1})$$
$$\# \ldots \underset{k_1+k_2+k_3}{\#} (\Sigma^{k_1+k_2+k_3}\times S^{k_4}\times\ldots\times S^{k_r}) \# \ldots \underset{n-k_r}{\#} (\Sigma^{n-k_r}\times S^{k_r})$$
$$\# \ldots \underset{n-k_1}{\#} (\Sigma^{n-k_1}\times S^{k_1})\Big] \# \Sigma^n \quad .$$

Hence the theorem.

Recall that $H(p,k)$ denotes the subgroup of θ^p consisting of homotopy p-spheres Σ^p such that $\Sigma^p \times S^p$ is diffeomorphic to $S^p \times S^k$.

THEOREM 3.2 The number of differentiable structures on $S^{k_1}\times\ldots\times S^{k_r}$ where $2 \le k_1 < k_2 < \ldots < k_{r-1}$ and $k_r-3 \le k_{r-1} \le k_r$ equals the order of the group

$$\frac{\theta^{k_1+k_r}}{H((k_1+k_r),(k_3,\ldots,k_{r-1}))} \times \frac{\theta^{k_2+k_r}}{H(k_2+k_r,(k_1,k_3,\ldots,k_{r-1}))} \times \ldots \times$$
$$\frac{\theta^{k_1+k_2+k_3}}{H((k_1+k_2+k_3,(k_4,\ldots,k_r))} \times \ldots \times \frac{\theta^{n-k_r}}{H(n-k_r,k_r)} \times \ldots \times \frac{\theta^{n-k_1}}{H(n-k_1,k_1)} \times \theta^n \quad .$$

PROOF Let $(0(k_1+k_r),0(k_2+k_r),\ldots,0(k_1+k_2+k_3),\ldots,0(n-k_r),\ldots,0(n-k_1),0(n))$ represent the trivial elements of $\theta^{k_1+k_r}, \theta^{k_2+k_r},\ldots,\theta^{k_1+k_2+k_3},\ldots,\theta^{n-k_1},\theta^n$, then we define a map

$$\beta:(\theta^{k_1+k_r}\times\theta^{k_2+k_r}\times\ldots\times\theta^{k_1+k_2+k_3}\times\ldots\times\theta^{n-k_r}\times\ldots\times\theta^{n-k_1}\times\theta^n, 0(k_1+k_r),\ldots,0(n-k_1),0(n))$$
$$\longrightarrow (\text{Structures on } S^{k_1}\times\ldots\times S^{k_r},0)$$

where 0 represents the usual structures on $S^{k_1}\times\ldots\times S^{k_r}$. If $\Sigma^{k_1+k_r} \in \theta^{k_1+k_r}$, $\ldots, \Sigma^{k_1+k_2+k_3} \in \theta^{k_1+k_2+k_3},\ldots, \Sigma^{n-k_r} \in \theta^{n-k_r},\ldots, \Sigma^{n-k_1} \in \theta^{n-k_1}$ and $\Sigma^n \in \theta^n$ then we define

$$\beta(\Sigma^{k_1+k_r}, \Sigma^{k_2+k_r},\ldots, \Sigma^{k_1+k_2+k_3},\ldots, \Sigma^{n-k_r},\ldots, \Sigma^{n-k_1}, \Sigma^n) =$$
$$\Big[(S^{k_1}\times\ldots\times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_r-1}) \underset{k_2+k_r}{\#} (\Sigma^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_r-1})$$
$$\# \ldots \underset{k_1+k_2+k_3}{\#} (\Sigma^{k_1+k_2+k_3}\times S^{k_4}\times\ldots\times S^{k_r}) \# \ldots \underset{n-k_r}{\#} (\Sigma^{n-k_r}\times S^{k_r})$$
$$\# \ldots \underset{n-k_1}{\#} (\Sigma^{n-k_1}\times S^{k_1})\Big] \# \Sigma^n \quad .$$

β is well-defined because if

$$\Sigma_1^{k_1+k_r}, \Sigma_2^{k_1+k_r} \in \theta^{k_1+k_r}; \Sigma_1^{k_2+k_r}, \Sigma_2^{k_2+k_r} \in \theta^{k_2+k_r}; \ldots \Sigma_1^{k_1+k_2+k_3}, \Sigma_2^{k_1+k_2+k_3} \in$$
$$\theta^{k_1+k_2+k_3}\ldots; \Sigma_1^{n-k_r}, \Sigma_2^{n-k_r} \in \theta^{n-k_r};\ldots; \Sigma_1^{n-k_1}, \Sigma_2^{n-k_1} \in \theta^{n-k_1}; \Sigma_1^n, \Sigma^n \in \theta^n$$

are h-cobordant respectively then they are diffeomorphic. It then follows that $\Sigma_1^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_r-1}$ is diffeomorphic to $\Sigma_2^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_r-1}$ and $\Sigma_1^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_r-1}$ is diffeomorphic to $\Sigma_2^{k_1+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_r-1}$ and

$\Sigma_1^{k_1+k_2+k_3} \times S^{k_4} \times S^{k_5} \times \ldots \times S^{k_r}$ is diffeomorphic to $\Sigma_2^{k_1+k_2+k_3} \times S^{k_4} \times \ldots \times S^{k_r}$. Also $\Sigma_1^{n-k_r} \times S^{k_r}$ is diffeomorphic to $\Sigma_2^{n-k_r} \times S^{k_r}$ and $\Sigma_1^{n-k_1} \times S^{k_1}$ is diffeomorphic to $\Sigma_2^{n-k_1} \times S^{k_1}$ and so this means that

$$\Big[(S^{k_1} \times \ldots \times S^{k_r}) \#_{k_1+k_r} (\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \#_{k_2+k_r} (\Sigma_1^{k_2+k_r} \times S^{k_3} \times \ldots \times S^{k_{r-1}}$$
$$\# \cdots \#_{k_1+k_2+k_3} (\Sigma_1^{k_1+k_2+k_3} \times S^{k_4} \times \ldots \times S^{k_r}) \# \ldots$$
$$\# \ldots \#_{n-k_r} (\Sigma_1^{n-k_r} \times S^{k_r}) \# \ldots \#_{n-k_1} (\Sigma_1^{n-k_1} \times S^{k_1}) \ \# \Sigma_1^n\Big] \text{ is diffeomorphic to}$$
$$\Big[(S^{k_1} \times \ldots \times S^{k_r}) \#_{k_1+k_r} (\Sigma_2^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \#_{k_2+k_r} (\Sigma_2^{k_2+k_r} \times S^{k_1} \times S^{k_3} \times \ldots \times S^{k_{r-1}})$$
$$\# \cdots \#_{k_1+k_2+k_3} (\Sigma_2^{k_1+k_2+k_3} \times S^{k_4} \times \ldots \times S^{k_r}) \# \ldots \#_{n-k_2} (\Sigma_2^{n-k_2} \times S^{k_r})$$
$$\# \ldots \#_{n-k_1} (\Sigma_2^{n-k_1} \times S^{k_1})\Big] \# \Sigma_2^n \ .$$

Hence β is well-defined map.

Clearly β takes the base points $0(k_1+k_r)$, $0(k_2+k_{r_1}), \ldots, 0(k_1+k_2+k_3), \ldots, 0(n-k_r), \ldots, 0(n-k_1)$, $0(n)$ to the base point 0. This is because if all the homotopy spheres Σ^ks are standard spheres, then all the summands involving Σ^is in the image of β will vanish leaving only $S^{k_1} \times \ldots \times S^{k_r}$. By Theorem 3.1, β is onto.

Suppose $\Sigma^{k_1+k_r} \in H((k_1+k_r),(k_2,\ldots,k_{r-1}))$, $\Sigma^{k_2+k_r} \in H((k_2+k_r),(k_1,k_3,\ldots,k_{r-1}))$, $\ldots$, $\Sigma^{k_1+k_2+k_3} \in H((k_1+k_2+k_3),(k_4,k_5,\ldots,k_r)), \ldots$, $\Sigma^{n-k_r} \in H(n-k_r,k_r), \ldots, \Sigma^{n-k_1} \in H(n-k_1,k_1)$ then for $\Sigma^{k_1+k_r} \in H((k_1+k_r),(k_2,\ldots,k_{r-1}))$ this means $\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}$ is diffeomorphic to $S^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}$ hence $S^{k_1} \times \ldots \times S^{k_r} \# \Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}$ is diffeomorphic to $(S^{k_1} \times \ldots \times S^{k_r}) \# S^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}} = S^{k_2} \times S^{k_3} \times \ldots \times S^{k_{r-1}} \times (S^{k_1} \times S^{k_r} \# S^{k_1+k_r})$ which is diffeomorphic to $S^{k_2} \times S^{k_3} \times \ldots \times S^{k_{r-1}} \times S^{k_1} \times S^{k_r}$ since $S^{k_1} \times S^{k_r} \# S^{k_1+k_r} = S^{k_1} \times S^{k_r}$. This means that for $\Sigma^{k_1+k_r} \in H((k_1+k_r),(k_2,\ldots,k_{r-1}))$ then $(S^{k_1} \times \ldots \times S^{k_r}) \#_{k_1+k_r} (\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}})$ is diffeomorphic to $S^{k_1} \times \ldots \times S^{k_r}$ and so the summand $\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}$ in the image β vanishes. Similar arguments show that all other summands involving the Σ^is in $H(i,n-i)$ vanish hence in this case

$$\beta(\Sigma^{k_1+k_r}, \Sigma^{k_2+k_r}, \ldots, \Sigma^{k_1+k_2+k_3}, \ldots, \Sigma^{n-k_r}, \ldots, \Sigma^{n-k_1}, \Sigma^n) = S^{k_1} \times S^{k_2} \times \ldots \times S^{k_r} \ .$$

Then β induces a map.

$$\Phi : \Big(\frac{\theta^{k_1+k_r}}{H(k_1+k_r,(k_2,\ldots,k_{r-1}))} \times \frac{\theta^{k_1+k_2}}{H(k_2+k_r,(k_1,k_2,\ldots,k_{r-1}))} \times \ldots \times$$
$$\frac{\theta^{k_1+k_2+k_3}}{H(k_1+k_2+k_3,(k_4,\ldots,k_r))} \times \ldots \times \frac{\theta^{n-k_r}}{H(n-k_r,k_r)} \times \ldots \times \frac{\theta^{n-k_1}}{H(n-k_1,k_1)}$$
$$\times\ \theta^n\Big) \longrightarrow (\text{structures on } S^{k_1} \times \ldots \times S^{k_r})$$

which is onto since β is onto.

If $\Phi(\Sigma^{k_1+k_r}, 0(k_2+k_r), \ldots, 0(k_1+k_2+k_3), \ldots, 0(n-k_r), \ldots, 0(n-k_1), 0(n)) = 0$ then it follows by an easy generalization of Theorem 2.2.1 of [1] that $\Sigma^{k_1+k_r} \in H((k_1+k_r),(k_2,\ldots,k_{r-1}))$ and by the same method if $\Phi(0(k_1+k_r), 0(k_2+k_r), \ldots, \Sigma^{n-k_r}), \ldots, \Sigma^{n-k_r}, \ldots, 0(n-k_1), 0(n)) = 0$ then $\Sigma^{n-k_r} \in H(n-k_r,k_r)$. Also in

[[5], Theorem A], Reinhard Schultz showed that the inertial group of product of any number of ordinary spheres is trivial. This result implies that if $\Phi(0(k_1+k_r), 0(k_2+k_r),\ldots,0(n-k_r),\ldots,0(n-k_1), \Sigma^n) = 0$ then Σ^n is diffeomorphic to S^n. It then follows that Φ is one to one and onto hence the number of differentiable structures on $S^{k_1}xS^{k_2}x\ldots xS^{k_r}$ is equal to the order of

$$\frac{\theta^{k_1+k_r}}{H(k_1+k_r,(k_2,\ldots,k_{r-1}))} x \frac{\theta^{k_2+k_r}}{H(k_2+k_r,k_1,k_3,\ldots,k_{r-1})} x \cdots x \frac{\theta^{k_1+k_2+k_3}}{H((k_1+k_2+k_3),(k_4,\ldots,k_r))}$$
$$\cdots x \frac{\theta^{k_1+k_2+k_3}}{H((k_1+k_2+k_3),(k_4,\ldots,k_r))} x \cdots x \frac{\theta^{n-k_1}}{H(n-k_r,k_r)} x \cdots x \frac{\theta^{n-k_1}}{H(n-k_1,k_1)} x \theta^n$$

EXAMPLES

We recall that in Table 7.4 of [5], θ_k^n denotes the number of homotopy spheres which do not embed in R^{n+k}. We shall use the values computed in that table in some of the examples given here. Since $\Gamma^i = 0$ for $1 \leq i \leq 6$, then the number of smooth structures on $S^2xS^2xS^2xS^2$ is the order of $\theta^8 = 2$. Also since $\theta_3^8 = 2 = |\theta^8|$ then $H(8,2) = 0$ and so the number of smooth structures on $S^2xS^2xS^2xS^2xS^2 = 12$. By similar reasoning, the number of smooth structures on $S^2xS^2xS^2xS^4 = 12$.

Since $\theta^{12} = 0$ and $H(9,3) = 4$ then the number of smooth structures on $S^3xS^3xS^3xS^3 = 2$ whereas since $\theta^{15} = 16256$ and $\theta^9 = 8$ combined with the fact that $\theta^{12} = 0$ and $H(9,3) = 4$ it follows that the number of smooth structures on $S^3xS^3xS^3xS^3xS^3$ is 32512. From [3] we see that $\theta_5^8 = 1$ and $H(8,4) = \theta^8$ and $\Gamma^{12}=0$, then the number of smooth structures on $S^4xS^4xS^4xS^4 = 2$. By a similar argument, it is easily seen that the number of smooth structures on $S^4xS^4xS^4xS^4xS^4$ is the order $\frac{\theta^{16}}{H(16,4)} x \theta^{20}$. Also since $H(10,5) = \theta^{10}$ then the number of smooth structures on $S^5xS^5xS^5xS^5$ is the order of $\frac{\theta^{15}}{H(15,5)} x \theta^{20}$.

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