DIFFERENTIABLE STRUCTURES ON A GENERALIZED PRODUCT OF SPHERES

SAMUEL OMOLYOE AJALA

School of Mathematics
The Institute for Advanced Study
Princeton, New Jersey 08540

and

Mathematics Department
University of Lagos
Akoka-Yaba
Lagos-Nigeria, West Africa

(Received April 21, 1986)

ABSTRACT. In this paper, we give a complete classification of smooth structures on a generalized product of spheres. The result generalizes our result in [1] and R. de Sapio's result in [2].

KEY WORDS AND PHRASES. Differential structures, product of spheres.

1980 AMS SUBJECT CLASSIFICATION CODE. 57R55

1. INTRODUCTION

In [2] a classification of smooth structures on product of spheres of the form $S^k \times S^p$ where $2 \leq k \leq p$, $k+p \geq 6$ was given by R. de Sapio and in [1] this author extended R. de Sapio's result to smooth structures on $S^p \times S^q \times S^r$ where $2 \leq p \leq q \leq r$. The next question is, how many differentiable structures are there in any arbitrary product of ordinary spheres. In this paper, we give a classification under the relation of orientation preserving diffeomorphism of all differentiable structures of $S^{k_1} \times S^{k_2} \times \ldots \times S^{k_r}$ where $2 \leq k_1 < k_2 < \ldots < k_r$. $S^n$ denotes the unit n-sphere with the usual differential structure in the Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$. $\Theta^n$ denotes the group of h-cobordism classes of homotopy n-sphere under the connected sum operation. $\Sigma^n$ will denote an homotopy n-sphere. $H(p,k)$ denotes the subset of $\Theta^n$ which consists of those homotopy p-sphere $\Sigma^p$ such that $\Sigma^p \times S^k$ is diffeomorphic to $S^p \times S^k$. By [2], $H(p,k)$ is a subgroup of $\Theta^n$ and it is not always zero and in fact in [1], we showed that if $k \geq p+3$, then $H(p,k) = \Theta^n$.

By Hauptreumung [3], piecewise linear homomorphism will be replaced by homeomorphism. Consider two manifolds $S^{k_1} \times S^{k_2} \times S^{k_3} \times S^{k_4}$ and $S^{k_2+k_4} \times S^{k_1} \times S^{k_3}$, we shall denote the connected sum of the two manifolds along a $k_2+k_4-1$ cycle by
removing \( \text{Int}(D^2 \times S) \times S \) from both manifolds and then identify their common boundary. Thus nothing else other than taking the usual connected sum of \( S^2 \times S \) and \( \Sigma^{k_2 + k_4} \) by removing the interior of an embedded disc \( D^2 \times S \) from each manifold and identify the manifolds along their common boundary \( S^2 \times S \times S \) to obtain \( S^2 \times S \times S \times S \). This is a well-defined operation. We then take the cartesian product with \( S^2 \times S \times S \) to have \( S^2 \times S \times S \times S \). But \( S^2 \times S \times S \times S \) is diffeomorphic to \( S^2 \times S \times S \times S \) then \( (S^2 \times S \times S \times S) \times \Sigma^{k_2 + k_4} \).

We will then prove the following.

**CLASSIFICATION THEOREM** If \( M^n \) is a smooth manifold homeomorphic to \( \Sigma^{k_1 + k_2 + \ldots + k_r} \) where \( 2 \leq k_1 < \ldots < k_r \) and \( k_4 - 3 < k < k_3 \) and \( n = k_1 + k_2 + \ldots + k_r \) then there exists homotopy spheres \( \Sigma^{k_1 + k_2 + \ldots + k_r} \times \ldots \times \Sigma^{k_1 + k_2 + \ldots + k_r} \) such that \( M^n \) is diffeomorphic to

\[
\left( \Sigma^{k_1 + k_2 + \ldots + k_r} \times \ldots \times \Sigma^{k_1 + k_2 + \ldots + k_r} \right)
\]

We shall use the above classification theorem to give the number of differentiable structures on \( S^2 \times S \times \ldots \times S \). We shall lastly compute the number of structures in some simple cases.

2. PRELIMINARY RESULTS

We shall apply obstruction theory of Munkres [4]. Let \( M \) and \( N \) be smooth \( n \)-manifolds and \( L \) a closed subset of \( M \) when triangulated. A homeomorphism \( f : M \to N \) is a diffeomorphism modulo \( L \) if \( f|_{(M-L)} \) is a diffeomorphism and each simplex \( \alpha \) of \( L \) has a neighborhood \( V \), such that \( f \) is smooth on \( V-L \) near \( \alpha \).

By [4], if two \( n \)-manifolds \( M \) and \( N \) are combinatorially equivalent then \( M \) is diffeomorphic modulo an \((n-1)\)-skeleton \( L \) onto \( N \).

If \( f : M^n \to N^n \) is a diffeomorphism modulo \( m \)-skeleton \( m < n \) then Munkres showed that the obstruction to deforming \( f \) to a diffeomorphism \( g : M^n \to N^n \) modulo \((m-1)\)-skeleton is an element \( \lambda_m(f) \in H_m(M,\Gamma^{n-m}) \). We call \( g \) the smoothing of \( f \). If \( \lambda_m(f) = 0 \) then \( g \) exists. Recall that in ([1], Lemma 2.1.1) we proved that if \( q > p \) then \( \Sigma^p \times S^q \) is diffeomorphic to \( S^q \times S^q \) for any homotopy sphere \( \Sigma^q \). In Remark (i) following that lemma, we showed further that even when \( p-3 \leq q \) the result is still true.

**LEMMA 2.1** Suppose \( f : M^n \to S^q \times S^q \times \ldots \times S^q \) is a piecewise linear homeomorphism which is a diffeomorphism modulo \((n-k_i)\)-skeleton \( 1 \leq i \leq r \), then there exists an
DIFFERENTIABLE STRUCTURES ON A GENERALIZED PRODUCT OF SPHERES

homotopy sphere $\Sigma^{k_i}$ and a piecewise linear homeomorphism

$$h: M^n \rightarrow S^{k_1} \times \ldots \times S^{k_r}$$

which is a diffeomorphism modulo $(n-k_i-1)$ skeleton.

**Proof.** Since $f: M^n \rightarrow S^{k_1} \times \ldots \times S^{k_r}$ is a diffeomorphism modulo $(n-k_i)$-skeleton then by Munkres [4], the obstruction to deforming $f$ to a diffeomorphism modulo $(n-k_i-1)$-skeleton is an element $\lambda_k(f) \in H_{n-k_i}(\mathbb{R}^n, \Gamma_{k_i}) = \Gamma_{k_i}$. Let

$$\psi = \chi_{k_i}^{-1} \chi_{k_i-1}$$

where $\chi: S^{k_i} \rightarrow S^{k_i}$ is a diffeomorphism. We define $k_i = \Sigma = \Sigma^{k_i} \cup \Sigma^{k_i-1}$ and so $j$ is identity map on $\text{Int}(\Sigma^{k_i})$ and radial extension of $\psi$ on $k_i$. So $j$ is a piecewise linear homeomorphism by the definition and the obstruction to deforming $j$ to a diffeomorphism is $[\psi^{-1}] = -\lambda_k(f)$. So consider the map

$$\text{id} \times j : (S^{k_1} \times \ldots \times S^{k_r}) \times (S^{k_1} \times \ldots \times S^{k_r}) \rightarrow (S^{k_1} \times \ldots \times S^{k_r}) \times (S^{k_1} \times \ldots \times S^{k_r})$$

The map is a piecewise linear homeomorphism and the obstruction to deforming it to a diffeomorphism is $[\psi^{-1}] = -\lambda_k(f)$. Notice that the manifold $(S^{k_1} \times \ldots \times S^{k_r}) \times (S^{k_1} \times \ldots \times S^{k_r})$ and $(S^{k_1} \times \ldots \times S^{k_r})$ are diffeomorphic modulo $(n-k_i-1)$ skeleton. Hence the lemma.

**Lemma 2.2** Let $f: M^n \rightarrow S^{k_1} \times \ldots \times S^{k_r}$ be a diffeomorphism modulo $n-(k_i+k_j)$ skeleton $1 \leq i,j \leq r$ then there exists homotopy sphere $\Sigma^{k_i+k_j}$ and a piecewise linear homeomorphism

$$f: M^n \rightarrow (S^{k_1} \times \ldots \times S^{k_r}) \# (\Sigma^{k_i+k_j} \times (S^{1} \times \ldots \times S^{k_i+k_j}))$$

which is a diffeomorphism modulo $n-(k_i+k_j)-1$ skeleton.

**Proof.** Since $f$ is a diffeomorphism modulo $n-(k_i+k_j)$ skeleton, it follows that the obstruction to deforming $f$ to a diffeomorphism modulo $n-(k_i+k_j)-1$ skeleton is

$$\lambda(f) \in H_{n-(k_i+k_j)}(\mathbb{R}^n, \Gamma_{k_i+k_j}) = \Gamma_{k_i+k_j}$$

where $\lambda(f) \in H_{n-(k_i+k_j)}(\mathbb{R}^n, \Gamma_{k_i+k_j})$. Let $[\chi] = \lambda(f) \in \Gamma_{k_i+k_j}$.

We define $j: S^{k_1} \times \ldots \times S^{k_r} \rightarrow S^{k_1} \times \ldots \times S^{k_r} \# \Sigma^{k_i+k_j} \times (S^{1} \times \ldots \times S^{k_i+k_j})$ to be identity map on $S^{k_1} \times \ldots \times S^{k_r} \# \text{Int}(\Sigma^{k_i+k_j})$ and radial extension of $\chi$ on $\text{Int}(\Sigma^{k_i+k_j})$ hence $j$ is a piecewise linear homeomorphism and the obstruction to deforming $j$ to a diffeomorphism is $[\chi^{-1}] = -\lambda(f)$. Then consider
Note that

\[\text{and} \quad k_1 x S^1 \times \ldots \times S^r \longrightarrow (S^1 \times S^1 \times \ldots \times S^r) \]

hence the above map is

\[\text{id} \times j: (S^1 \times \ldots \times S^r) \longrightarrow (S^1 \times \ldots \times S^r) \]

which is piecewise linear and its obstruction to a diffeomorphism is \(-\lambda(f)\) hence the obstruction to deforming the composite \((j \times \text{id}) \circ f\) to a diffeomorphism modulo \(n-(k_1+k_r)-1\) skeleton is zero. Hence if \(f' = (j \times \text{id}) \circ f\) then \(f': M^n \longrightarrow M^n\) is a diffeomorphism modulo \(n-(k_1+k_r)-1\) skeleton.

3. CLASSIFICATION

**THEOREM 3.1** If \(M^n\) is a smooth manifold homeomorphic to \(S^1 \times S^2 \times \ldots \times S^r\) then there exists homotopy spheres, \(\Sigma^1, \Sigma^2, \ldots, \Sigma^r, \ldots, \Sigma^n\), and \(\Sigma^n\) such that \(M^n\) is diffeomorphic to

\[\text{where} \quad 2 \leq k_1 < k_2 < \ldots < k_r, \quad k_r \geq k_1 + k_2 + \ldots + k_r.

**PROOF.** Suppose \(M^n \stackrel{h}{\longrightarrow} S^1 \times \ldots \times S^r\) is the homeomorphism. By Munkres theory [4], \(h\) is a diffeomorphism modulo \((n-1)\) skeleton. Since the first non-zero homology appears in dimension \(n-k_1\), \(h\) is a diffeomorphism modulo \((n-k_1)\) skeleton. The obstruction to deforming \(h\) to a diffeomorphism modulo \((n-k_1)\) skeleton is \(\lambda(h) \in H_{n-k_1} (M^n, \Gamma^1) = \Gamma^1\). By Lemma 2.1, there exists a piecewise linear homeomorphism \(h'\) and a homotopy sphere \(\Sigma^1\) such that \(h' : M^n \rightarrow \Sigma^1 \times S^1 \times \ldots \times S^r\) which is a diffeomorphism modulo \((n-k_1-1)\)
In Lemma 2.1.1 it was proved that $\Sigma_{k_1 \times S^{k_2}}$ is diffeomorphic to $S^{k_1 \times S^{k_2}}$ since $k_1 < k_2$. It then follows that $\Sigma_{k_1 \times S^{k_2}} \times S^{k_3} \times \ldots \times S^{k_r}$ is diffeomorphic to $S^{k_1 \times S^{k_2}} \times S^{k_3} \times \ldots \times S^{k_r}$ hence $h' : M^n \to S^{k_1 \times S^{k_2}} \times S^{k_3} \times \ldots \times S^{k_r}$ is a diffeomorphism modulo $(n-k_1-1)$ skeleton. There is no other obstruction to deforming $h'$ to a diffeomorphism until the $(n-k_2-1)$-skeleton. This is because

$$H_i(M^n, \mathbb{Z}) = 0$$

for $n-k_2+1 < i < n-k_1$. So we can assume that $h'$ is a diffeomorphism modulo $(n-k_2-1)$ skeleton. The obstruction to deforming $h'$ to a diffeomorphism modulo $(n-k_2-1)$ skeleton is $\lambda(h') \in H_{n-k_2} (M^n, \mathbb{Z}^{k_2}) = \mathbb{Z}^{k_2}$. Again by Lemma 2.1, there exists a homotopy sphere $\Sigma_{k_2}$ and a piecewise linear homeomorphism $h'' : M^n \to S^{k_1 \times S^{k_2}} \times S^{k_3} \times \ldots \times S^{k_r}$ which is a diffeomorphism modulo $(n-k_2-1)$ skeleton. By the same argument as above since $k_2 < k_3$ we see that $\Sigma_{k_2 \times S^{k_3}}$ is diffeomorphic to $S^{k_1 \times S^{k_2}} \times S^{k_3} \times \ldots \times S^{k_r}$ hence $S^{k_1 \times S^{k_2}} \times S^{k_3} \times \ldots \times S^{k_r}$ is diffeomorphic to $S^{k_1 \times S^{k_3}} \times S^{k_2} \times \ldots \times S^{k_r}$. This shows that $h'' : M^n \to S^{k_1 \times S^{k_2}} \times S^{k_3} \times \ldots \times S^{k_r}$ is a diffeomorphism modulo $(n-k_2-1)$-skeleton. By the same argument since $M^n$ has no homology between $n-k_3-1$ and $n-k_2-1$ we can assume that $h''$ is a diffeomorphism modulo $(n-k_3)$-skeleton. Proceeding this way using the same argument we can construct a homeomorphism say $h''' : M^n \to S^{k_1 \times S^{k_2}} \times S^{k_3} \times \ldots \times S^{k_r}$ which is a diffeomorphism modulo $(n-k_r)$-skeleton. However, to deform $h'''$ to a diffeomorphism modulo $(n-k_r-1)$-skeleton, there is an obstruction $\lambda(h''') \in H_{n-k_r} (M^n, \mathbb{Z}^{k_r}) = \mathbb{Z}^{k_r}$. It also follows by Lemma 2.1 that there exists a homotopy sphere $\Sigma_{k_r}$ and a piecewise linear homeomorphism $f : M^n \to S^{k_1 \times S^{k_2}} \times S^{k_3} \times \ldots \times S^{k_r}$ which is a diffeomorphism modulo $(n-k_r)$-skeleton. Now in Remark (1) of [1] it was shown that even when $p-3 \leq r$, $S^r \times S^p$ is diffeomorphic to $S^r \times S^p$ and so by our assumption that $k_1-3 \leq k_r-1 \leq k_r$ it follows that $S^{k_r-1} \times S^{k_r}$ is diffeomorphic to $S^{k_r-1} \times S^{k_r}$. Hence $S^{k_1 \times S^{k_2} \times S^{k_3} \times \ldots \times S^{k_r-1} \times S^{k_r}}$ is diffeomorphic to $S^{k_1 \times S^{k_2} \times S^{k_3} \times \ldots \times S^{k_r-1} \times S^{k_r}}$ and so $f : M^n \to S^{k_1 \times S^{k_2} \times S^{k_3} \times \ldots \times S^{k_r}}$ is a diffeomorphism modulo $(n-k_r-1)$-skeleton. The next obstruction to deforming $f$ to a diffeomorphism is on the $(n-(k_1+k_r)-1)$ skeleton and it is $\lambda(f) \in H_{n-(k_1+k_r)} (M^n, \mathbb{Z}^{k_1+k_r}) = \mathbb{Z}^{k_1+k_r}$. By Lemma 2.2, there exists a piecewise linear homeomorphism $f' : M^n \to (S^{k_1 \times S^{k_2}} \times \ldots \times S^{k_r}) \setminus (\Sigma_k S^{k_1 \times S^{k_2} \times \ldots \times S^{k_r-1}})^{k_1+k_r}$ which is a diffeomorphism modulo $(n-(k_1+k_r)-1)$ skeleton for some homotopy sphere $S^{k_1+k_r}$ defined using $\lambda(f) \in \mathbb{Z}^{k_1+k_r}$. At this point, we want to remark that if $k_1 \geq k_r-3 \leq k_r$ and suppose $k_r = \text{max}(k_2, \ldots, k_r)$ then it follows from Remark (1) of [1] since $k_1 \geq k_r-3 \leq k_r$ that $S^{k_1+k_r} \times S^{k_j}$ is diffeomorphic to $S^{k_1+k_r} \times S^{k_1 \times S^{k_2} \times \ldots \times S^{k_r-1}}$ and so $S^{k_1+k_r} \times S^{k_1 \times S^{k_2} \times \ldots \times S^{k_r-1}}$ is diffeomorphic to $S^{k_1+k_r} \times S^{k_1 \times S^{k_2} \times \ldots \times S^{k_r-1}}$. This then implies that $S^{k_1 \times S^{k_2} \times \ldots \times S^{k_r-1}} \setminus (\Sigma_k S^{k_1 \times S^{k_2} \times \ldots \times S^{k_r-1}})^{k_1+k_r}$ is diffeomorphic to $(S^{k_1 \times S^{k_2} \times \ldots \times S^{k_r-1}})^{k_1+k_r} \setminus (\Sigma_k S^{k_1 \times S^{k_2} \times \ldots \times S^{k_r-1}})^{k_1+k_r}$ and this is diffeomorphic to $S^{k_1 \times S^{k_2} \times \ldots \times S^{k_r-1}}$ because $S^{k_1 \times S^{k_2} \times \ldots \times S^{k_r-1}}$. So this means that the factor
\[ \Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}} \] will disappear in the above sum if we have the condition
\[ k_1+3 \leq \max(k_2, \ldots, k_{r-1}). \]

Anyway, we have \( f': H^n \rightarrow (S^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \) which is a diffeomorphism modulo \( n-(k_1+k_r)-1 \) skeleton. Since \( H_i(M^n, \mathbb{Z}) = 0 \) for \( n-(k_1+k_r) < i \leq n-(k_1+k_r)-1 \) then there is no obstruction to deforming \( f' \) to a diffeomorphism modulo \( n-(k_2+k_r) \) skeleton and the obstruction to deforming \( f' \) to a diffeomorphism modulo \( n-(k_2+k_r)-1 \) skeleton is \( \lambda(f') \in H_{n-(k_2+k_r)}(M^n, \mathbb{Z}^{k_2+k_r}) \). Using the same technique as in the proof of Lemma 2.2 it can be easily shown that there exists an homotopy sphere \( \Sigma^{k_2+k_r} = \partial S^{k_2+k_r} \cup S^{k_2+k_r} \) where \( \psi = \lambda(f') \in \pi_{k_2+k_r}^{k_2+k_r} \) and
\[ \psi: S^{k_2+k_r-1} \rightarrow S^{k_2+k_r-1} \] is a diffeomorphism and a piecewise linear homeomorphism
\[ j: S^{k_1+k_r} \rightarrow (S^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \]

where obstruction to a diffeomorphism is \( -\lambda(f') \). We now define a map
\[ j': (S^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \]

\[ \rightarrow (S^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \]

where \( j' = j \) on \( (S^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \) and identity on \( \Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}} \). Clearly \( j' \) is piecewise linear and its obstruction to a diffeomorphism is \( -\lambda(f') \) hence the obstruction to deforming the composite \( g = j' \cdot f' \) where \( g: H^n \rightarrow (S^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \) is
\[ \lambda(j' \cdot f') = \lambda(j') + \lambda(f') = 0. \]
Hence \( g = j' \cdot f' \) is a diffeomorphism modulo \( n-(k_2+k_r)-1 \) skeleton. Proceeding in this way, we see that the next obstruction to a diffeomorphism will be on \( n-(k_3+k_r)-1 \) skeleton. Using the above technique continuously, we can construct a piecewise linear homeomorphism
\[ g': H^n \rightarrow (S^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \]

\[ \rightarrow (S^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \]

\[ \text{which is a diffeomorphism modulo } n-(k_{r-1}+\ldots+k_1) = k_r \text{ skeleton. The obstruction to extending } g' \text{ to a diffeomorphism modulo } (k_r-1) \text{ skeleton is } \lambda(g') \in H_{k_r}(H^n, \mathbb{Z}^{k_r}) = \pi_{k_r}^{k_r}. \]

By using the same technique as in the proof of Lemma 2.1, there exists a piecewise linear homeomorphism \( j \) and homotopy sphere \( \Sigma^{n-k_r} \) such that
\[ j: S^{n-k_r} \rightarrow (S^{n-k_r} \times S^{k_r}) \]

\[ \rightarrow (S^{n-k_r} \times S^{k_r}) \]

has an obstruction to a diffeomorphism to be \( -\lambda(g') \). From this we define the map,
where \( j' = j \) on \((S^{k_1} \times \ldots \times S^{k_r}) \cap (\text{Int}(D^{k_1+k_r}) \times S^{k_2} \times \ldots \times S^{k_{r-1}})\) and identity elsewhere. It is easily seen that \( j' \) is piecewise linear homeomorphism and the obstruction to deforming the composite \( j' \cdot g' \) to a diffeomorphism is zero. Hence the map \( h' = j' \cdot g' \) where

\[
h' : M^n \to (S^{k_1} \times \ldots \times S^{k_r}) \cap (\text{Int}(D^{k_1+k_r}) \times S^{k_2} \times \ldots \times S^{k_{r-1}})
\]

is a diffeomorphism modulo \((k_r-1)\) skeleton. However, since \( H_i(M^n, \mathbb{Z}) = 0 \) for \( k_{r-1} < i < k_r - 1 \), there is no more obstruction to deforming \( h' \) to a diffeomorphism modulo \( k_{r-1} \)-skeleton. To deform \( h' \) to a diffeomorphism modulo \((k_{r-1}-1)\) skeleton, there is an obstruction and this equals \( \lambda(h') \in H_{k_{r-1}}(M^n, \mathbb{Z}) = \Gamma^{n-k_{r-1}} \). Applying the above technique again, we can get an homotopy sphere \( \Sigma^{n-k_{r-1}} \) and a piecewise linear homeomorphism

\[
h'' : M^n \to (S^{k_1} \times \ldots \times S^{k_r}) \cap (\text{Int}(D^{k_1+k_r}) \times S^{k_2} \times \ldots \times S^{k_{r-1}})
\]

which is a diffeomorphism modulo \((k_{r-1}-1)\) skeleton. The next obstruction will be on \( k_{r-2} \)-skeleton. Proceeding this way gradually down the remaining skeleton, we can construct a map

\[
g : M^n \to (S^{k_1} \times \ldots \times S^{k_r}) \cap (\text{Int}(D^{k_1+k_r}) \times S^{k_2} \times \ldots \times S^{k_{r-1}})
\]

which is a diffeomorphism modulo \( k_1 \)-skeleton. Since \( H_i(M^n, \mathbb{Z}) = 0 \) for \( 0 < i < k \), then \( g \) is a diffeomorphism modulo one point. It therefore follows that there exist an homotopy sphere \( \Sigma^n \) such that \( M^n \) is diffeomorphic to
Recall that $H(p,k)$ denotes the subgroup of $\Theta^p$ consisting of homotopy $p$-spheres $\Sigma^p$ such that $\Sigma^p \times S^p$ is diffeomorphic to $S^p \times S^k$. 

**Theorem 3.2** The number of differentiable structures on $S^{k_1} \times \ldots \times S^{k_r}$ where $2 \leq k_1 < k_2 < \ldots < k_{r-1}$ and $k_r \geq k_{r-1} \geq k_r$ equals the order of the group $H((k_1 + k_r), (k_2, \ldots, k_r-1)) \times \ldots \times H((k_1 + k_r), (k_2, \ldots, k_r-1)) \times \ldots \times H((k_1 + k_r), (k_2, \ldots, k_r-1)) \times \ldots \times H((k_1 + k_r), (k_2, \ldots, k_r-1)) \times \Theta^n$.

**Proof** Let $(0(k_1 + k_r), 0(k_2 + k_r), \ldots, 0(k_r + k_r), 0(n-k_1), \ldots, 0(n-k_r), 0(n))$ represent the trivial elements of $\Theta^1 = 0(k_1 + k_r), \Theta^2 = 0(k_2 + k_r), \ldots, \Theta^{n-k_1} = 0(n-k_1), \Theta^n$, then we define a map

$$\beta : (\Theta^1, \Theta^2, \ldots, \Theta^{n-k_1}, \Theta^n) \longrightarrow (\text{Structures on } S^{k_1} \times \ldots \times S^{k_r}, 0)$$

where 0 represents the usual structures on $S^{k_1} \times \ldots \times S^{k_r}$. If $\Sigma^1 + k_r \in \Theta^1, \ldots, \Sigma^{n-k_1} \in \Theta^{n-k_1}, \Sigma^n \in \Theta^n$ then we define

$$\beta((\Sigma^1, \ldots, \Sigma^n), (\Theta^1, \ldots, \Theta^n)) = \left((\Theta^1, \ldots, \Theta^n) \times (\Sigma^1, \ldots, \Sigma^n)\right)$$

$\beta$ is well-defined because if

$$\Sigma^1, \Sigma^2, \ldots, \Sigma^n \in \Theta$$

are h-cobordant respectively then they are diffeomorphic. It then follows that $\Sigma^1 \times \ldots \times \Sigma^n$ is diffeomorphic to $\Sigma^1 \times \ldots \times \Sigma^n$ and

$$\Sigma^1 \times \ldots \times \Sigma^n$$
Differentiable structures on a generalized product of spheres

\[ \sum^{k_1+k_2+k_3} x^{k_4} \times x^{k_5} \ldots \times x^{k_r} \] is diffeomorphic to \[ \sum^{k_1+k_2+k_3} x^{k_4} \times x^{k_5} \ldots \times x^{k_r} \]. Also \[ \Sigma^{-k_r} x^{k_r} \] is diffeomorphic to \[ \Sigma^{-n-k_r} x^{k_r} \] and \[ \Sigma^{-n-k_1} x^{k_1} \] is diffeomorphic to \[ \Sigma^{-n-k_2} x^{k_2} \] and so this means that

\[ (s^{k_1} x \ldots x s^{k_r}) \] \[ \# \Sigma^{k_1+k_2+k_3} x^{k_4} \times x^{k_5} \ldots \times x^{k_r} \] \[ \# \ldots \# \Sigma^{n-k_1 x^{k_1}} \] \[ \ldots \# \Sigma^{n-k_2 x^{k_2}} \] \[ \# \ldots \# \Sigma^{n-k_n} \] is diffeomorphic to

\[ (s^{k_1} x \ldots x s^{k_r}) \] \[ \# \Sigma^{k_1+k_2+k_3} x^{k_4} \times x^{k_5} \ldots \times x^{k_r} \] \[ \# \ldots \# \Sigma^{n-k_1 x^{k_1}} \] \[ \ldots \# \Sigma^{n-k_2 x^{k_2}} \] \[ \# \ldots \# \Sigma^{n-k_n} \] .

Hence \( \beta \) is well-defined map.

Clearly \( \beta \) takes the base points \( 0(k_1+k_r), 0(k_2+k_{r-1}), \ldots, 0(k_1+k_2+k_3), \ldots, 0(n-k_1), \ldots, 0(n-k_r), 0(n) \) to the base point \( 0 \). This is because if all the homotopy spheres \( s^{k_i} \) are standard spheres, then all the summands involving \( s^{k_i} \) in the image of \( \beta \) will vanish leaving only \( s^{k_1} x \ldots x s^{k_r} \). By Theorem 3.1, \( \beta \) is onto.

Suppose \( \Sigma^{k_1+k_r} \in H((k_1+k_r), (k_2, \ldots, k_{r-1})) \), \( \Sigma^{k_1+k_2+k_3} \in H((k_1+k_2+k_3), (k_4, \ldots, k_r)) \), \( \ldots, \Sigma^{k_1+k_2+k_3} \in H((k_1+k_2+k_3), (k_4, \ldots, k_r)) \), \( \ldots, \Sigma^{n-k_1 x^{k_1}} \in H((n-k_1), k_1) \), then for \( \Sigma^{k_1+k_r} \in H((k_1+k_r), (k_2, \ldots, k_{r-1})) \) this means \( \Sigma^{k_1+k_2+k_3} x^{k_4} \ldots x^{k_r-1} \) is diffeomorphic to \( s^{k_1+k_2+k_3} x^{k_4} \ldots x^{k_r-1} \) hence \( s^{k_1+x} \ldots x s^{k_r} \) \[ \# \Sigma^{k_1+k_2+k_3} x^{k_4} \ldots x^{k_r-1} \] is diffeomorphic to \[ (s^{k_1} x \ldots x s^{k_r}) \] \[ \# s^{k_1+k_2+k_3} x^{k_4} \ldots x^{k_r-1} = s^{k_2} x s^{k_3} \ldots x s^{k_r-1} x (s^{k_1} x^{k_1} \# s^{k_1+k_r}) \] which is diffeomorphic to \( s^{k_2} x s^{k_3} \ldots x s^{k_r-1} \) since \( s^{k_1+x} \# s^{k_1+k_r} = s^{k_1+x} \) .

This means that for \( \Sigma^{k_1+k_r} \in H((k_1+k_r), (k_2, \ldots, k_{r-1})) \) then \( s^{k_1} x \ldots x s^{k_r} \) \[ \# (s^{k_1+k_r} x^{k_2} \ldots x s^{k_r-1}) \] is diffeomorphic to \( s^{k_1+x} \ldots x s^{k_r} \) and so the summand \( \Sigma^{k_1+k_r} x^{k_2} \ldots x s^{k_r-1} \) in the image \( \beta \) vanishes. Similar arguments show that all other summands involving the \( s^{k_i} \) in \( H(i, n-i) \) vanish hence in this case
\[ \beta(\Sigma^{k_1+k_r}, \ldots, \Sigma^{k_2+k_3}, \ldots, \Sigma^{n-k_r}, \ldots, \Sigma^{n-k_1}, \Sigma^{n}) = s^{k_1+x} s^{k_2} \ldots x s^{k_r} \].

Then \( \beta \) induces a map
\[ \beta : \begin{pmatrix} \begin{pmatrix} \Sigma^{k_1+k_r} \\ H(k_1+k_r) x^{k_2} x^{k_3} \ldots x^{k_r-1} \end{pmatrix} \times \begin{pmatrix} \Sigma^{k_2+k_3} \\ H(k_1+k_2+k_3) x^{k_4} x^{k_5} \ldots x^{k_r} \end{pmatrix} \times \ldots \times \begin{pmatrix} \Sigma^{n-k_r} \\ H(n-k_r) \end{pmatrix} \times \begin{pmatrix} \Sigma^{n-k_1} \\ H(n-k_1, k_1) \end{pmatrix} \end{pmatrix} \times \begin{pmatrix} \Sigma^{n} \end{pmatrix} \rightarrow \text{(structures on } s^{k_1+x} x^{k_2} x^{k_3} \ldots x^{k_r}) \]
which is onto since \( \beta \) is onto.

If \( \beta(\Sigma^{k_1+k_r}, 0(k_2+k_r), \ldots, 0(k_1+k_2+k_3), \ldots, 0(n-k_r), \ldots, 0(n-k_1), 0(n)) = 0 \) then it follows by an easy generalization of Theorem 2.2.1 of [1] that \( \Sigma^{k_1+k_2+k_3} \in H((k_1+k_r), (k_2, \ldots, k_{r-1})) \) and by the same method if \( \beta(0(k_1+k_r), 0(k_2+k_r), \ldots, 0(n-k_1), 0(n)) = 0 \) then \( \Sigma^{n-k_r} \in H(n-k_r, k_r) \). Also in
Reinhard Schultz showed that the inertial group of product of any number of ordinary spheres is trivial. This result implies that if \( \Sigma(0(k_1+k_r), (0(k_1+k_r),\ldots,0(n-k_r),\ldots,0(n-k_1),\Sigma^r) = 0 \) then \( \Sigma^n \) is diffeomorphic to \( S^n \). It then follows that \( \theta \) is one to one and onto hence the number of differentiable structures on \( S^{k_1+k_2+\ldots+k_r} \) is equal to the order of

\[
\frac{\theta^{k_1+k_2+k_3}}{\theta^{k_1+k_2+k_3}} \times \frac{\theta^{k_2+k_3}}{\theta^{k_2+k_3}} \times \ldots \times \frac{\theta^{n-k_1}}{\theta^{n-k_1}} \times \frac{\theta^{k_1+k_2+k_3}}{\theta^{k_1+k_2+k_3}} \times \frac{\theta^{k_2+k_3}}{\theta^{k_2+k_3}} \times \ldots \times \frac{\theta^{n-k_1}}{\theta^{n-k_1}} \times \theta^n
\]

**EXAMPLES**

We recall that in Table 7.4 of [5], \( \theta^n \) denotes the number of homotopy spheres which do not embed in \( R^{n+k} \). We shall use the values computed in that table in some of the examples given here. Since \( \theta^i = 0 \) for \( 1 \leq i \leq 6 \), then the number of smooth structures on \( S^2 \times S^2 \times S^2 \times S^2 \) is the order of \( \theta^8 = 2 \). Also since \( \theta^8 = 2 = |\theta^8| \) then \( H(8,2) = 0 \) and so the number of smooth structures on \( S^2 \times S^2 \times S^2 \times S^2 = 12 \). By similar reasoning, the number of smooth structures on \( S^4 \times S^2 \times S^2 \times S^4 = 12 \).

Since \( \theta^{12} = 0 \) and \( H(9,3) = 4 \) then the number of smooth structures on \( S^3 \times S^3 \times S^3 \times S^3 = 2 \) whereas since \( \theta^{15} = 16256 \) and \( \theta^9 = 8 \) combined with the fact that \( \theta^{12} = 0 \) and \( H(9,3) = 4 \) it follows that the number of smooth structures on \( S^3 \times S^3 \times S^3 \times S^3 \) is 32512. From [3] we see that \( \theta^8 = 2 \) and \( H(8,4) = 0 \) and \( \theta^{10} = 0 \), then the number of smooth structures on \( S^4 \times S^4 \times S^4 \times S^4 = 2 \). By a similar argument, it is easy to see that the number of smooth structures on \( S^4 \times S^4 \times S^4 \times S^4 \) is the order \( \frac{\theta^{15}}{H(15,5) \times \theta^{20}} \). Also since \( H(10,5) = \theta^{10} \) then the number of smooth structures on \( S^5 \times S^5 \times S^5 \) is the order of \( \frac{\theta^{15}}{H(15,5) \times \theta^{20}} \).

I am grateful to Professor P. Emery Thomas, University of California, Berkeley, for very useful communications.

**REFERENCES**

Submit your manuscripts at
http://www.hindawi.com