UNSTABLE PERIODIC WAVE SOLUTIONS OF NERVE AXON DIFFUSION EQUATIONS

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ABSTRACT. Unstable periodic solutions of systems of parabolic equations are studied. Special attention is given to the existence and stability of solutions.

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1. INTRODUCTION.

Diffusion systems of partial differential equations are of great importance in biosciences. In this paper, unstable periodic solutions of systems of the form

\[ u_t = u_{xx} + F(u, w), \]
\[ w_t = G(u, w), \]  

are studied. Equations of this type arise in neurophysiology in the study of nerve impulses on nerve axon, see [1,2]. Other classes of diffusion equations are also involved in biology, see for example [3-9].

2. EXISTENCE OF SOLUTIONS

It is known that for \( G(u, w) = \varepsilon u \), if \( \varepsilon > 0 \) is sufficiently small, equation (1.1) has two types of wave solutions, namely, pulse travelling wave solutions and periodic travelling wave solutions. A travelling wave solution is a solution of equation (1.1) of the form

\[ [u(x, t), w(x, t)] = [\phi(z; c), \psi(x; c)], \]

hence \( [\phi(z; c), \psi(z; c)] \) satisfies the ordinary differential equation

\[ \frac{d^2 \phi}{dz^2} - c \frac{d\phi}{dz} + F(\phi, \psi) = 0, \]  
\[ - c \frac{d\psi}{dz} + G(\phi, \psi) = 0. \]  

A pulse travelling wave solution is a non-constant solution of (2.1) satisfying
and a periodic travelling wave solution is a periodic solution of (2.1).

In [10], Evans showed that equation (1.1) has two pulse travelling solutions with different propagation speeds \( c_1 \) and \( c_2 \). On the existence of periodic travelling wave solutions, Hastings [11] showed that equation (2.1) with \( G(u,w) = cu \) has a non-constant periodic solution if \( c > 0 \) is sufficiently small and the speed \( c \) is limited to a certain range. Rinzel and Keller [12] studied the case in which \( F(u,w) \) is a function of \( u \) only given by

\[
F(u,w) = \begin{cases} 
  u & \text{for } u \leq a, \\
  u-1 & \text{for } a < u,
\end{cases}
\]

where \( 0 < a < 1/2 \). Under this assumption, equation (2.1) has a non-constant periodic solution if \( c \) is limited in the range \( c_1 < c < c_2 \) and the period \( P(c) \) is a smooth function of \( c \). They demonstrated the behavior of the function \( P(c) \) under the two cases when \( a \) is not very small and when \( a \) is very small. Dai [13] proved the existence and uniqueness of solutions for a general case and studied stability of the solution.

3. STABILITY ANALYSIS.

Stability of periodic travelling wave solutions is related to the eigenvalues of a matrix in the following theorem. Let \( A(z;\lambda,c) \) be the matrix

\[
A(z;\lambda,c) = \begin{bmatrix}
  0 & 1 & 0 \\
 -F_1(\Phi(z;c),\Psi(z;c)) & c & -F_2(\Phi(z;c),\Psi(z;c)) \\
 G_1(\Phi(z;c),\Psi(z;c)) & 0 & G_2(\Phi(z;c),\Psi(z;c)) - \lambda
\end{bmatrix}
\]

where \( F_1 \) and \( G_1 \) denote the partial derivatives as usual, and let \( X(z;\lambda,c) \) be a matrix satisfying the differential equation

\[
\frac{d}{dz} X = A X
\]

with the initial condition \( X(0;\lambda,c) = I \).

**Theorem 3.1.** Suppose the functions \( F \) and \( G \) in equation (1.1) satisfy (a) \( F(0,0) = 0 \), (b) \( G(0,0) = 0 \) and (c) the matrix \( X(p(c);\lambda,c) \) has an eigenvalue of modulus 1, for some complex number \( \lambda \) with \( \text{Re} \lambda > 0 \), then a periodic travelling wave solution \( [\Phi(z;c),\Psi(z;c)] \) is unstable.

**Proof.** With the change of variables,

\[
z = x + ct, \\
t = t,
\]

\( [u(x,t), w(x,t)] = [\widetilde{u}(z,t), \widetilde{w}(z,t)] \),
equation (1.1) becomes

$$\tilde{u}_t = \tilde{u}_{zz} - c \tilde{u}_z + F(\tilde{u}, \tilde{w}),$$  \hspace{1cm} (3.1)$$

$$\tilde{w}_t = -c \tilde{w}_z + G(\tilde{u}, \tilde{w}).$$

The linearized perturbation equation of the above system with respect to the solution \([\phi(z;c), \psi(z;c)]\) is

$$\tilde{U}_t = \tilde{U}_{zz} - c \tilde{U}_z + F_1 [\phi, \psi] \tilde{U} + F_2 [\phi, \psi] \tilde{W},$$
$$\tilde{W}_t = -c \tilde{W}_z + G_1 [\phi, \psi] \tilde{U} + G_2 [\phi, \psi] \tilde{W},$$  \hspace{1cm} (3.2)$$

where \(\phi = \phi(z;c)\) and \(\psi = \psi(z;c)\), since \(F(0,0) = G(0,0) = 0\). Equation (3.2) has a solution of the form

$$\tilde{U}(z,t) = e^{\lambda t} y_1 (z;\lambda),$$
$$\tilde{W}(z,t) = e^{\lambda t} y_2 (z;\lambda),$$

where \((y_1, y_2)\) satisfies the following system of linear ordinary differential equations

$$\lambda y_1 = \frac{d^2 y_1}{dz^2} - c \frac{dy_1}{dz} + F_1 [\phi, \psi] y_1 + F_2 [\phi, \psi] y_2,$$
$$\lambda y_2 = -c \frac{dy_2}{dz} + G_1 [\phi, \psi] y_1 + G_2 [\phi, \psi] y_2,$$  \hspace{1cm} (3.3)$$

where \(\phi = \phi(z;c)\) and \(\psi = \psi(z;c)\). Note that if equation (3.3) has a solution which is bounded for all \(z\) in \((-\infty, \infty)\) for a number \(\lambda\) with \(\text{Re}(\lambda) > 0\), then equation (3.2) has a solution \([\tilde{U}(z,t), \tilde{W}(z,t)]\) which grows exponentially, and hence, the travelling wave solution \([\phi(z;c), \psi(z;c)]\) is unstable.

Using Floquet's theory, we can show that equation (3.3) has a bounded non-trivial solution if and only if one of the eigenvalues of \(X(p(c);\lambda, c)\) is a modulus 1. Equation (3.3) can be rewritten as

$$\frac{d}{dz} \left( \frac{dy_1}{dz} \right) = \left( \lambda - F_1 [\phi, \psi] \right) y_1 + c \frac{dy_1}{dz} - F_2 [\phi, \psi] y_2,$$
$$c \frac{dy_2}{dz} = G_1 [\phi, \psi] y_1 + (G_2 [\phi, \psi] - \lambda) y_2,$$

and so can be represented by the matrix differential equation

$$\frac{d}{dz} \mathbf{y} = A(z;\lambda, c) \mathbf{y},$$
where

$$
\begin{bmatrix}
    y_1 \\
    \frac{dy_1}{dz} \\
    y_2
\end{bmatrix}
$$

and the matrix $A$ is as defined before. Now, since the coefficient matrix $A(z;\lambda,c)$ is a $p(c)$-periodic function of $z$, Floquet's theory yields that equation (3.3) has a bounded non-trivial solution if and only if one of the eigenvalues of the matrix $X(p(c);\lambda,c)$ defined before is of modulus 1. The proof is now complete.

In the following lemma, it is shown that under the special case $\lambda = 0$, one eigenvalue of $X(p(c);0,c)$ is unity and the product of the other two eigenvalues is greater than one.

**Lemma 3.1.** Suppose (a) $G_2(u,w) \geq 0$ for all $u$ and $w$ and (b) $\lambda = 0$, let $\mu_i(\lambda,c), i = 1, 2, 3$, denote the eigenvalues of $X(p(c);\lambda,c)$, then one eigenvalue, say

$$
\mu_1(0,c) = 1,
$$

and

$$
\mu_2(0,c)\mu_3(0,c) > 1.
$$

**Proof.** Differentiation of equation (2.1) leads to

$$
\frac{d}{dz} \left( \frac{d^2 \phi}{dz^2} \right) = c \frac{d}{dz} \left( \frac{d\phi}{dz} \right) - F_1[\phi,\psi] \frac{d\phi}{dz} - F_2[\phi,\psi] \frac{d\psi}{dz} - G_1[\phi,\psi] \frac{d\phi}{dz} - G_2[\phi,\psi] \frac{d\psi}{dz},
$$

(3.4)

where $\phi = \phi(z;c)$ and $\psi = \psi(z;c)$. Therefore the vector

$$
\begin{bmatrix}
    \phi \\
    \frac{d\phi}{dz} \\
    \psi
\end{bmatrix}
$$

satisfies the matrix equation

$$
\frac{d}{dz} w = A(z;0,c) w,
$$

that is,

$$
\begin{bmatrix}
    \frac{d\phi}{dz} \\
    \frac{d^2 \phi}{dz^2} \\
    \frac{d\psi}{dz}
\end{bmatrix}
= \begin{bmatrix}
    0 & 1 & 0 \\
    -F_1[\phi(z;c), \psi(z;c)] & c & -F_2[\phi(z;c), \psi(z;c)] \\
    G_1[\phi(z;c), \psi(z;c)] & 0 & G_2[\phi(z;c), \psi(z;c)]
\end{bmatrix}
\begin{bmatrix}
    \frac{d\phi}{dz} \\
    \frac{d^2 \phi}{dz^2} \\
    \frac{d\psi}{dz}
\end{bmatrix}.
$$
We know that (see for example, Sanchez)

$$\frac{w_z}{z}(z;c) = X(z;0,c) w_z(0;c)$$

and since $w_z(z;c)$ is a $p(c)$-periodic function of $z$, it follows that

$$\frac{w_z}{z}(0;c) = w_z(p(c);c) = X(p(c);0,c) \frac{w_z}{z}(0;c). \tag{3.5}$$

Thus there is an eigenvalue, say

$$\mu_1(0,c) = 1.$$

Further, by Jacobi's formula,

$$\det \{X(z;\lambda,c)\} = \det X(0;\lambda,c) \exp \int_0^z \text{tr} \{A(\xi;\lambda,c)\} \, d\xi$$

$$= (1) \exp \int_0^z \left( c \frac{G_2[\phi,\psi]}{c} - \lambda \right) \, d\xi.$$

In particular,

$$\det \{X(p(c);0,c)\} = \exp \left[ c \int_0^{p(c)} \frac{G_2[\phi,\psi]}{c} \, d\xi \right] > 1$$

since $c > 0$, $p(c) > 0$ and $G_2(u,w) \geq 0$ for all $u,w$.

But $\det \{X(p(c);0,c)\} = \mu_1(0,c) \mu_2(0,c) \mu_3(0,c)$ and

$$\mu_1(0,c) = 1, \text{ hence } \mu_2(0,c) \mu_3(0,c) > 1.$$

Note that under the assumptions of Lemma 3.1, either $|\mu_2(\lambda,c)| > 1$ or $|\mu_3(\lambda,c)| > 1$ for $\lambda$ sufficiently small. In the next theorem, we will see that if $L(c)$ is decreasing, i.e. $L'(c) < 0$, then $\mu_1(\lambda,c)$ is increasing at $\lambda = 0$, i.e. $\frac{\partial}{\partial \lambda} \mid_{\lambda=0} \mu_1(\lambda,c) > 0$.

**Theorem 3.2.** Suppose (a) $p'(c) < 0$, then $\frac{\partial}{\partial \lambda} \mid_{\lambda=0} \mu_1(\lambda,c) > 0$, and hence if (b) the assumptions in Lemma 3.1 also hold, then $\mu_1(\lambda,c) > 1$ for $\lambda$ sufficiently small.

**Proof:** We claim that the following equality

$$\frac{\partial}{\partial \lambda} \mu_1(\lambda,c) \mid_{\lambda=0} = -p'(c)$$

actually holds.
Recall the vector \( \mathbf{w}(z;c) \), namely,

\[
\mathbf{w} = \begin{bmatrix} \phi \\ \frac{d\phi}{dz} \\ \psi \\ \frac{d\psi}{dz} \end{bmatrix}
\]

which satisfies the periodicity

\( \mathbf{w}(p(c);c) = \mathbf{w}(0;c) \).

Differentiation of the above equation with respect to \( c \) leads to

\[
\frac{\partial}{\partial z} \mathbf{w}(p(c);c) \mathbf{p}'(c) + \mathbf{w}_c(p(c);c) = \frac{\partial}{\partial c} \mathbf{w}(0;c). \quad (3.6)
\]

Let \( \mathbf{v} = [\mathbf{v}_1^*(z;\lambda,c), \mathbf{v}_2^*(z;\lambda,c)] \) be a solution of equation (3.3) satisfying the initial condition

\[
\mathbf{v}(0;\lambda,c) = \mathbf{w}_z(0;c) + \lambda \mathbf{w}_c(0;c), \quad (3.7)
\]

where \( \mathbf{v}(z;\lambda,c) \) is the vector defined before. We have observed before that \( \mathbf{v}_1(z;c), \mathbf{v}_2(z;c) \), which satisfies equation (3.4), is a solution of equation (3.3) under \( \lambda = 0 \). In view of the condition (3.7) and by uniqueness of solutions, we have

\[
\mathbf{v}^*(z;0,c) = \mathbf{w}_z(z;c). \quad (3.8)
\]

Differentiation of equation (3.3) with respect to \( \lambda \) leads to

\[
y_1 + \lambda \frac{\partial y_1}{\partial \lambda} = \frac{d^2}{dz^2} \left( \frac{\partial y_1}{\partial \lambda} \right) - c \frac{d}{dz} \left( \frac{\partial y_1}{\partial \lambda} \right) + F_1(\phi,\psi) \frac{\partial y_1}{\partial \lambda} + F_2(\phi,\psi) \frac{\partial y_2}{\partial \lambda},
\]

\[
y_2 + \lambda \frac{\partial y_2}{\partial \lambda} = -c \frac{d}{dz} \left( \frac{\partial y_2}{\partial \lambda} \right) + G_1(\phi,\psi) \frac{\partial y_1}{\partial \lambda} + G_2(\phi,\psi) \frac{\partial y_2}{\partial \lambda}. \quad (3.9)
\]

Under \( \lambda = 0 \), and replacing \( [y_1, y_2] \) by \( [y_1^*, y_2^*] \), equation (3.9) by equality (3.8) becomes

\[
\frac{d\phi}{dz}(z;c) = \frac{d^2}{dz^2} \left( \frac{\partial y_1^*}{\partial \lambda} \right) - c \frac{d}{dz} \left( \frac{\partial y_1^*}{\partial \lambda} \right) + F_1(\phi,\psi) \frac{\partial y_1^*}{\partial \lambda} + F_2(\phi,\psi) \frac{\partial y_2^*}{\partial \lambda},
\]

\[
\frac{d\psi}{dz}(z;c) = -c \frac{d}{dz} \left( \frac{\partial y_2^*}{\partial \lambda} \right) + G_1(\phi,\psi) \frac{\partial y_1^*}{\partial \lambda} + G_2(\phi,\psi) \frac{\partial y_2^*}{\partial \lambda}, \quad (3.10)
\]

where \( \frac{\partial y_1^*}{\partial \lambda} = \frac{\partial y_2^*}{\partial \lambda} \) now. On the other hand, differentiating equation (2.1) with respect to \( c \), we get
\[
\frac{d^2}{dz^2} \left( \frac{3y_1}{3c} \right) - c \frac{d}{dz} \left( \frac{3y_2}{3c} \right) + F_1[\phi, \psi] \frac{3y_1}{3c} + F_2[\phi, \psi] \frac{3y_2}{3c} = 0,
\]
\[
- \frac{d\phi}{dz} - c \frac{d}{dz} \left( \frac{3y_1}{3c} \right) + G_1[\phi, \psi] \frac{3y_1}{3c} + G_2[\phi, \psi] \frac{3y_2}{3c} = 0,
\]
where \( \phi = \phi(z; c) \) and \( \psi = \psi(z; c) \). Therefore both \( \frac{d}{dy_1}(z;0;c), \frac{d}{dy_2}(z;0;c) \) and \( \frac{d\phi}{dz}(z;c), \frac{d\psi}{dz}(z;c) \) satisfy the same differential equation. In addition, differentiation of the initial condition (3.7) yields
\[
\nu_\lambda(0;\lambda,c) = \omega_c(0;c),
\]
in particular,
\[
\nu_\lambda(0;0,c) = \omega_c(0;c)
\]
and hence the equality
\[
\nu_\lambda^*(z;0,c) = \omega_c(z;c), \quad 0 \leq z \leq p(c).
\]
The equalities (3.8) and (3.12) together give
\[
\nu_\lambda^*(z;\lambda,c) = \omega_c(z;c) - \lambda \omega_c(z;c) + o(\lambda^2), \quad 0 \leq z \leq p(c), \quad \lambda \to 0.
\]
Knowing \( \nu_\lambda^*(z;\lambda,c) = X(z;\lambda,c) \nu_\lambda^*(0;\lambda,c) \), by equation (3.13) for \( z = p(c) \) and also \( z = 0 \), we get
\[
\omega_c(p(c);c) + \lambda \omega_c(p(c);c) + o(\lambda^2)
\]
\[
= X(p(c);\lambda,c) [\omega_c(0;c) + \lambda \omega_c(0;c)].
\]
Substitution of the equation (3.6) containing \( p'(c) \) into the left hand side of equation (3.14) and periodicity lead to
\[
X(p(c);\lambda,c) [\omega_c(0;c) + \lambda \omega_c(0;c)]
\]
\[
= [1 - \lambda p'(c)][\omega_c(0;c) + \lambda \omega_c(0;c)] + o(\lambda^2).
\]
Hence the eigenvalue \( \nu_\lambda^*(\lambda,c) \) satisfies
\[
\frac{d}{dx} \nu_\lambda^*(\lambda,c) \bigg|_{\lambda=0} = -p'(c).
\]
The proof is now complete.
On the other hand, under certain conditions, two eigenvalues have modulus less than one and one has modulus greater than one.

**Theorem 3.3.** Suppose (a) $F_2(u,w)$ is a non-zero constant and (b) $G_1(u,w)$ and $G_2(u,w)$ are constant, then for $\lambda$ sufficiently large, two eigenvalues of $X(p(c);\lambda,c)$ have modulus $< 1$ and one has modulus $> 1$.

**Proof:** Decompose the matrix $A(z;\lambda,c)$ as follows

$$A(z;\lambda,c) = B(\lambda,c) + E(z;c)$$

where

$$B(\lambda,c) = \begin{bmatrix}
0 & 1 & 0 \\
\lambda & c & -F_2 \\
G_1 & 0 & G_2 - \lambda/c
\end{bmatrix}$$

and

$$E(z;c) = \begin{bmatrix}
0 & 0 & 0 \\
F_1(\psi(z;c), \psi(z;c)) & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.$$

Let $s_i(\lambda,c)$, $i = 1, 2, 3$ be the eigenvalues of $B(\lambda,c)$ and $q_i$ the corresponding eigenvectors. The characteristic equation of $B(\lambda,c)$ is

$$-s^3 + \left(\frac{G_2 - \lambda}{c} + c\right)s^2 + (2\lambda - G_2)s + \lambda - G_2 - \frac{F_2 G_1}{c} = 0.$$ 

It follows that as $\lambda \to \infty$,

$$s_1(\lambda,c) = \frac{-\lambda}{c} + o(1)$$

$$s_2(\lambda,c) = -\sqrt{\lambda} + o(1) \quad (3.15)$$

$$s_3(\lambda,c) = \sqrt{\lambda} + o(1).$$

The vectors $q_i(\lambda,c)$ are

$$q_i(\lambda,c) = \begin{bmatrix}
1 \\
s_i \\
s_i^2 - c_s_i - \lambda \\
-F_2
\end{bmatrix}, \quad i = 1, 2, 3,$$ \hspace{1cm} (3.16)

and let $Q(\lambda,c)$ be the non-singular matrix

$$Q(\lambda,c) = [q_1(\lambda,c), q_2(\lambda,c), q_3(\lambda,c)],$$

then

$$Q^{-1}BQ = \begin{bmatrix}
s_1(\lambda,c) & 0 & 0 \\
0 & s_2(\lambda,v) & 0 \\
0 & 0 & s_3(\lambda,c)
\end{bmatrix}.$$
Now consider the matrix

$$Y(z;\lambda,c) = Q^{-1} X(z;\lambda,c) Q$$

which has the same eigenvalues as $X(z;\lambda,c)$, in particular with $z = \rho(c)$, and satisfies the differential equation

$$\frac{d}{dz} Y(z;\lambda,c) = Q^{-1} A(z;\lambda,c) Q Y(z;\lambda,c)$$

$$= [Q^{-1} B(\lambda,c) Q + Q^{-1} E(z;c) Q] Y(z;\lambda,c),$$

since $\frac{d}{dz} X(z;\lambda,c) = A(z;\lambda,c) X(z;\lambda,c)$.

But $Q^{-1} BQ$ is the diagonal matrix from before and it can be shown easily using (3.15) and (3.16) that all elements of $Q^{-1} BQ$ are $o(1)$ as $\lambda \to \infty$, therefore the eigenvalues of $Y(\rho(c);\lambda,c)$ and hence of $X(\rho(c);\lambda,c)$ approach

$$\exp \left[ s_i(\lambda,c) \rho(c) \right], \ i = 1,2,3 \text{ as } \lambda \to \infty.$$ 

It follows from (3.15) that as $\lambda \to \infty$, two eigenvalues of $X(\rho(c);\lambda,c)$ have modulus $< 1$ and one has modulus $> 1$.

To summarize, under the assumptions of both Theorem (3.2) and Theorem (3.3), at least two eigenvalues of $X(\rho(c);\lambda,c)$ have modulus $> 1$ as $\lambda \to 0^+$, and two eigenvalues of $X(\rho(c);\lambda,c)$ have modulus $< 1$ as $\lambda \to \infty$. Hence one of the eigenvalues must have modulus $= 1$ for some $\lambda > 0$ and under Theorem (3.1), the travelling wave solution $(\Phi(z;c), \Psi(z;c))$ is unstable.

REFERENCES


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