ON COMPLEX L_1 -PREDUAL SPACES

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ABSTRACT - This paper contains some characterisations of complex L_1 -predual spaces which are being known as Lindenstrauss spaces after [1]. The dual unit balls of such spaces now-a-days called L -balls have been characterised by many authors including Lazar & Lindenstrauss [2], Lazar [3], [4], Lau [5] and others when the spaces are real. But their complex versions far from being trivial follow-ups seem to be much complicated and in reality sometimes need ingenuity to be formulated even. This paper contains some complex versions of Lau's results [5] embodied in Theorem 3.

KEY WORDS AND PHRASES: *Boundary measures, Choquet ordering, Haar measure, Lindenstrauss space, Simplex.*

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1. INTRODUCTION.

The purpose of this paper is to furnish some characterisations of complex L_1 -predual spaces which are being known as Lindenstrauss spaces after [1]. The dual unit balls of such spaces now-a-days called L -balls have been characterised by many authors including Lazar & Lindenstrauss [2], Lazar [3], [4], Lau [5] and others when the spaces are real. But their complex versions far from being trivial follow-ups seem to be much complicated and in reality sometimes need ingenuity to be formulated even. This paper contains some complex versions of Lau's results [5] embodied in Theorem 3.

2. NOTATIONS AND PRELIMINARIES

For a compact convex subset K of a locally convex Hausdorff space E , $\partial_e K$ stands for the set of its extreme points; $M(K)$ for the Banach space (with total variation as norm) of complex regular Borel measures on K ; $M_1(K)$ for the set of members of $M(K)$ with norm ≤ 1 ; $C(K)$, $A(K)$, $P(K)$ for the space of all real-valued continuous functions, continuous affine functions, continuous convex functions on K respectively.

For bounded real-valued functions f on K , the upper envelope is denoted by $\hat{f}$ and the lower envelope by $\check{f}$. A measure μ is said to be a boundary measure if $|\mu|$ is maximal in the ordering of Choquet; in fact μ is a boundary measure iff $\mu(\hat{f}-f)=0$ for all $f \in C(K)$ [6; p.129]. We shall also write $\Gamma = \{z \in \mathbb{C}: |z|=1\}$.

If V is a complex Banach space, the dual unit ball $K=(V^*)_1$ is convex and compact in the w^* -topology. We define the map $\text{hom } f$ as $(\text{hom } f)(x) = \int_{\Gamma} \bar{\alpha} f(\alpha x) d\alpha$ for semi-continuous function f on K , where $d\alpha$ is the unit Haar measure on Γ . Clearly $\text{hom } f$ is Γ -homogeneous, i.e. $(\text{hom } f)(\beta x) = \beta(\text{hom } f)(x)$ for $\beta \in \Gamma$. One can easily show that hom restricted to $C(K)$ are norm-decreasing projections of $C(K)$ onto the space of Γ -homogeneous

continuous functions on K . The adjoint projection hom defined as $\text{hom } \mu = \mu \circ \text{hom}$ is also a norm decreasing w^* -continuous projection of $M(K)$ onto a linear subspace

$M_{\text{hom}}(K)$ of $M(K)$. We can write $(\text{hom } f)(x) = S_1 f(x) + i T f(x)$

$$\text{where } S_1 f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos \theta f(xe^{i\theta}) d\theta, T f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin \theta f(xe^{i\theta}) d\theta$$

$$\text{If we write } (Sf)(x) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \cos \theta f(xe^{i\theta}) d\theta$$

(which multiplied by π gives what Roy [7] has defined as f_+), then

$$2(S_1 f)(x) = \text{odd } Sf(x). \quad (2.1)$$

Throughout the paper we shall write $A_0(K)$ for the set of all continuous Γ -homogeneous affine functions on $K=(V^*)_1$.

3. Main Results

For real Banach space V , the following results are recently proved.

Theorem 1. If K is the dual unit ball of a real Banach space V , then the following are equivalent:

- (i) V is an L_1 -predual space.
- (ii) If μ_1, μ_2 are boundary measures on K having the same barycentre, then $\text{odd } \mu_1 = \text{odd } \mu_2$.
- (iii) For $f \in P(K)$, $\hat{\text{odd}} f$ is affine.
- (iv) For any $f \in P(K)$, $\hat{f}(0) = \frac{1}{2} \sup\{f(x) + f(-x) : x \in K\} = \sup\{\text{even } f(x) : x \in K\}$.
- (v) For any l.s.c. concave function f on K such that $\text{even } f \geq 0$, there exists a continuous affine symmetric function a on K such that $f \geq a$.
- (vi) If $f, -h$ are l.s.c. concave functions on K such that $h \leq f$ and $\sup\{\text{even } h(x) : x \in K\} \leq \inf\{\text{even } f(x) : x \in K\}$, then there exists a continuous affine symmetric function a on K such that $h \leq a \leq f$.

The equivalence of (i) - (iv) is due to Lazar [4] while that of (i), (v), (vi) is proved by Lau [5].

Many interesting developments are noted when efforts are made to obtain complex analogs of these results (many others not stated here) of real Lindenstrauss spaces. A brilliant step towards this have been made by Effros [8] who has shown that $\text{odd } \mu$ is to be replaced by $\text{hom } \mu$ in complex space. Olsen [9] has shown that the hypothesis $\text{even } f \geq 0$ in (v) is to be replaced by $\sum f(\zeta_k x) \geq 0$ for $\zeta_k \in \Gamma$ with $\sum \zeta_k = 0$, $x \in K$. Subsequently Roy [7] has tried to give complex analog of (iv), replacing $\hat{\text{odd}} f$ by $\hat{\text{odd}}(Sf)$. His formulation is rather partial. But [9] contains some interesting examples.

Below we give a characterisation of complex L_1 -predual space V which is a kind of complex analog of Lau's result and is due to Olsen [9].

Theorem 2. If K is the dual unit ball of a complex Banach space V , then the following are equivalent:

- (i) V is an L_1 -predual space;
- (ii) For every l.s.c. concave function f on K such that $\sum f(\zeta_k x) \geq 0$ for all $x \in K$ and $\zeta_k \in \Gamma$, $k = 1, 2, \dots, n$ with $\sum \zeta_k = 0$, there is an $a \in A_0(K)$ such that $\text{re } a(x) \leq f(x)$ for all $x \in K$.

We give in Theorem 3, some complex analogs of Lau's result. However to start with, we furnish a Lemma below:

Lemma 1. If μ be a non-zero positive measure on a compact convex subset K of a locally convex Hausdorff space E , then for all u.s.c. convex function f on K

$$f(x) \leq \mu(K)^{-1} \int f(y) d\mu \quad \text{where } r(\mu) = x.$$

Proof: By a well-known result [10; I.2.2.], the stated inequality holds for $f \in P(K)$. Now applying Mokobodzki ([10; I.5.1]) that for every u.s.c. convex function f , there is a descending net $\{f_\alpha : f_\alpha \in P(K)\}$ which converges to f , we get the desired result.

Our main result is

Theorem 3. If K is the dual unit ball of a complex Banach space V , then the following are equivalent:

- (i) V is an L_1 -predual space;
- (ii) If f is a l.s.c. concave function on K with even $Sf(x) \geq 0$ for all $x \in K$, then there exists an $a \in A_0(K)$ such that $re a \leq f$ on K ;
- (iii) If $f, -h$ are l.s.c. concave functions on K such that $h \leq f$ and even $Sh(x) \leq 0 \leq \text{even } Sf(y)$ for all $x, y \in K$; then there exists an $a \in A_0(K)$ such that $h \leq re a \leq f$ on K .
- (iv) If $f, -h$ are l.s.c. concave functions on K such that $h \leq f$ and
$$\sup_{k=1}^n \{ \sum_{k=1}^n \alpha_k h(\zeta_k x) : x \in K, n \in \mathbb{N}, 0 \leq \alpha_k, \sum \alpha_k = 1, \zeta_k \in \Gamma, \sum \alpha_k \zeta_k = 0 \}$$
$$\leq 0 \leq \inf_{k=1}^n \{ \sum_{k=1}^n \alpha_k f(\zeta_k x) : x \in K, n \in \mathbb{N}, 0 \leq \alpha_k, \sum \alpha_k = 1, \zeta_k \in \Gamma, \sum \alpha_k \zeta_k = 0 \};$$
 then there is an $a \in A_0(K)$ such that $h \leq re a \leq f$ on K ;
- (v) If g is an u.s.c. convex function on K , then $\hat{g}(0) \leq \sup \{ \sum \alpha_k g(\zeta_k x) : x \in K, n \in \mathbb{N}, 0 \leq \alpha_k, \sum \alpha_k = 1, \zeta_k \in \Gamma, \sum \alpha_k \zeta_k = 0 \}$.

Proof . (i) $\rightarrow$ (ii).

We shall, in fact, show that (ii) is implied by Theorem 2 (ii). So let f be l.s.c. concave on K such that even $Sf(x) \geq 0$. We define

$$F(x) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \left[\cos \theta f(xe^{i\theta}) \right] d\theta$$

Then $F(x) = 2 Sf(x)$. Clearly F is l.s.c. concave. Let $\zeta_k \in \Gamma$ for $k = 1, 2, \dots, n$ be such that $\sum \zeta_k = 0$. Now note that $Sf(x) = S_1 f(x) + \text{even } Sf(x)$ and that $\sum \text{Hom } f(\zeta_k x) = 0$. Thus $\sum F(\zeta_k x) = 2 \sum Sf(\zeta_k x) = 2 \sum S_1 f(\zeta_k x) + 2 \sum \text{even } Sf(\zeta_k x) = 2 \sum \text{even } Sf(\zeta_k x)$ which is ≥ 0 by hypothesis. Consequently by Theorem 2 (ii), there is a function $b \in A_0(K)$ such that $re b \leq F$ on K .

We consider the measure
$$\mu = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \cos \theta \varepsilon_{(xe^{i\theta})} d\theta$$

where $\varepsilon(y)$ is the Dirac measure at y . By [6: p. 115], $r(\mu) = x$. Also $\mu(K) = \frac{4}{\pi}$. On applying Lemma 1, we have $2F(x) \leq \mu(K)f(x)$ i.e. $F(x) \leq \frac{2}{\pi} f(x)$. Putting $a = \frac{\pi b}{2}$, we have $re a \leq f$ on K .

(ii) $\rightarrow$ (iii). Let $f, -h$ be l.s.c. concave functions on K such that $h \leq f$ and even $S h(x) \leq 0 \leq S f(y)$ for all $x, y \in K$.

We first show that $h(x) + h(-x) \leq 0 \leq f(y) + f(-y)$ for all $x, y \in K$. Let us establish the last inequality, since the first one can be done similarly. To do so we take the measure

$$\mu = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \cos \theta \, \epsilon(xe^{i\theta}) d\theta$$

and find as before by [6, p. 115] that $r(\mu) = x$, $\mu(K) = \frac{4}{\pi}$.

Now apply Lemma 1 to get $f(y) + f(-y) \geq 0$ from even $S f(y) \geq 0$.

Now we define $F(x) = f(x) \wedge (-h)(-x)$.

Then F is l.s.c. concave. Moreover by the hypothesis and the inequalities just proved, we have $F(x) + F(-x) \geq 0$ for all $x \in K$ so that even $S F(x) \geq 0$. By (ii) then we have an $a \in A_0(K)$ such that $re a \leq F$. This a is, in fact, the function with the desired property.

(iii) $\rightarrow$ (iv). Let $f, -h$ be two l.s.c. concave functions on K , which satisfy the conditions given in the hypothesis of (iv). Then clearly even $(Sh)(x)$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \cos \theta [h(xe^{i\theta}) + h(-xe^{i\theta})] d\theta \leq 0$$

Similarly even $(Sf)(y) \geq 0$ for all $y \in K$.

So by (iii) there is an $a \in A_0(K)$ such that $h \leq re a \leq f$ on K .

(iv) $\xrightarrow{n}$ (v). Suppose that g is an u.s.c. convex function and let $g_0 = \sup \{ \sum_{k=1}^n \alpha_k g(\zeta_k x) : x \in K; n \in \mathbb{N}; 0 \leq \alpha_k; \sum \alpha_k = 1, \zeta_k \in \Gamma, \sum \alpha_k \zeta_k = 0 \}$.

We assume that $g_0 = 0$; there will be no loss of generality in the assumption since $(g + \alpha)^\wedge(0) = \hat{g}(0) + \alpha$

for positive real number α .

We define $F = -\sigma g$ where $(\sigma g)(x) = g(-x)$. Clearly F is l.s.c. concave.

Since $g + \sigma g \leq 2g_0$ by the definition of g_0 , it follows that $g \leq F$. Moreover for $n \in \mathbb{N}$, $\alpha_k \geq 0$, $\sum \alpha_k = 1$, $\zeta_k \in \Gamma$, $\sum \alpha_k \zeta_k = 0$, $x \in K$, we have $\sum \alpha_k F(\zeta_k x) = -\sum \alpha_k g(-\zeta_k x)$

so that $\inf \{ \sum_{k=1}^n \alpha_k F(\zeta_k x) : x \in K, n \in \mathbb{N}, \alpha_k \geq 0, \sum \alpha_k = 1, \zeta_k \in \Gamma, \sum \alpha_k \zeta_k = 0 \} = -g_0 = 0$.

Thus by (iv), there is an $h \in A_0(K)$ such that $g \leq re h \leq F$. We put $re h = a \in A(K)$. Now since $g \leq a$, we have $\hat{g}(0) \leq a(0)$. Again $-\sigma g \geq a - 2g_0 \in A(K)$, so that $a(0) - 2g_0 \leq (-g)(0) = -\hat{g}(0)$.

Thus $\hat{g}(0) \leq a(0) \leq -\hat{g}(0) + 2g_0$ so that $\hat{g}(0) \leq g_0$ and the result follows.

(v) $\rightarrow$ (i) is the same as [7; p. 103]

Note: Our result in (v) is sharper than Roy's result [7; Thm 3.3 (iii)] that for $g \in P(K)$, $\hat{g}(0) \leq \sup \{ \sum \alpha_k g(\zeta_k x) : x \in K, n \in \mathbb{N}, 0 \leq \alpha_k, \sum \alpha_k = 1, \zeta_k \in \Gamma, \sum \alpha_k \zeta_k = 0 \}$.

In fact this follows from (v) immediately, since the reverse inequality is evident from the concave character of $\hat{g}$.

REMARKS:

In analog with Lazar's selection theorem for Choquet simplex, Lazar & Lindenstrauss [2] formulated a selection theorem for real L -balls which was followed by a complex version by Olsen [9]. Our results which are chiefly complex analogue of Lau's result [3] seem to resemble Edward's interpolation theorem [10; [II.3-10]] for simplices.

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