AN INVERSE PROBLEM FOR HELMHOLTZ'S EQUATION

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ABSTRACT. The refraction coefficient in Helmholtz's equation is found from the knowledge of a family of the solutions to this equation on two lines.

KEYS WORDS AND PHRASES. Helmholtz equation, inverse problem, Born approximation.

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1. INTRODUCTION.

Let

\[ [\nabla^2 + k^2 v(x)]u = -\delta(x-y), \quad k > 0 \tag{1.1} \]

where \( x = (x_1, x_3), y = (y_1, y_3), v = v(x_1, x_3), u = u(x_1, x_3, y_1, y_3, k) \).

Assume that

\[ v(x) = 0 \text{ for } x_1 \geq a \text{ or } x_3 \leq -a, \text{ or } x_3 \geq 0 \text{ or } x_3 \leq -R, v \in L^2 \tag{1.2} \]

Here \( R > 0 \) is an arbitrary large fixed number. Write (1.1) as

\[ u = g + k^2 \int g v d\mathbf{z}, \quad g := (i/4) H_0^{(1)}(k|x-y|) \tag{1.3} \]

where the integral is taken over the support of \( v \) and \( H_0^{(1)} \) is the Hankel function.

The problem is: find \( v(k) \) from the knowledge of \( u(-a, x_3, a, y_3, k) \) for all \( -\infty < x_3, y_3 < \infty \) and \( 0 < k < k_0 \), where \( k_0 > 0 \) is an arbitrary small number.

2. SOLUTION.

Let \( L = \{ x: x_1 = a, x_3 \in \mathbb{R} \} \), \( R = (-\infty, \infty) \). We use the method given in [1], [2]. It follows from (1.3) that

\[ f(x_3, y_3, k) := k^2(u-g) = \int g v d\mathbf{z} + o(k) \quad \text{as } k \to 0, \quad x \in L - a, \quad y \in L - a. \tag{2.1} \]

Let us take the Fourier transform of (2.1) in \( x_3 \) and \( y_3 \), define

\[ \hat{f}(\lambda, \mu) := (2\pi)^{-2} \int_{-\infty}^{\infty} \exp(-i\lambda x_3 - i\mu y_3) f(x_3, y_3) d\lambda d\mu, \quad \text{and use the formula} \]

\[ (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-i\lambda x_3) g(x, z) dx_3 = i(4\pi)^{-1} \exp\left(-i\lambda z_3 + i(\alpha + z_1)(k^2 - \lambda^2)^{1/2}\right)/(k^2 - \lambda^2)^{1/2} \tag{2.2} \]
where \( x = (-a, x_3) \), the radical \((k^2 + \lambda^2)^{1/2} > 0\) for \( \lambda^2 < k^2 \) and is defined by analytic continuation for all complex \( \lambda \) on the complex \( \lambda \)-plane with the cut \((-k, k)\), \( k > 0 \), so that
\[
(k^2 - \lambda^2)^{1/2} = i (\lambda^2 - k^2)^{1/2} \quad \text{if} \quad k^2 < \lambda^2.
\] (2.3)

The result is
\[
\tilde{f}(\lambda, \mu) = \int dz v(z) h(\lambda, \mu, z, k) + o(k)
\] (2.4)
where for \( k^2 > \lambda^2, \ k^2 > \mu^2 \), and \( r(\lambda) := (k^2 - \lambda^2)^{1/2} \) one has
\[
h := - (16\pi^2)^{-1} \exp \{ -i(\lambda + \mu)z_3 + i(\alpha + z_1) r(\lambda) - i(\alpha - z_1) r(\mu) \} r^{-1}(\lambda) r^{-1}(\mu)
\] (2.5)
and for \( k^2 < \mu^2 \) and \( k^2 < \lambda^2 \) one uses (2.3).

In the Born approximation one drops the term \( o(k) \) in (2.4) and solves the resulting linear integral equation for \( v(z) \) [2].

In the exact theory one passed to the limit \( k \to 0 \) in (2.4), obtains a linear integral equation for \( v \) and solves this equation analytically [2]. It is not possible to pass to the limit \( k \to 0 \) in (2.1) because \( g(kr) = a(k) + g_0 + o((kr)^2 \ln(k/2)) \) as \( k \to 0 \), where \( g_0 := (2\pi)^{-1} \ln(r^{-1}) \), \( a(k) := - (2\pi)^{-1} \ln(k/2) + i\gamma/(2\pi) \), and \( \gamma = 0.5572 \ldots \) is Euler's constant. Thus \( g(kr) \) does not have a finite limit as \( k \to 0 \). Nevertheless one can pass to the limit \( k \to 0 \) in (2.4) if \( \gamma \neq 0 \) or \( \mu \neq 0 \). The reason is that the term \( a(k) \) in (2.1) after the Fourier transform becomes \( a(k) \delta(\lambda) \delta(\mu) \), and this term, which contains the factor \( a(k) = \infty \) as \( k \to 0 \), is zero for \( \lambda \neq 0 \) or \( \mu \neq 0 \). Another way to study the limit behavior of the solution to (2.1) is given in [2]. To give the exact theory, pass to the limit \( k \to 0 \) in (2.4) to get
\[
\int v(z) \exp(-ipz_3 + qz_1) dz_1 dz_3 = \psi(p, q)
\] (2.6)
where we used (2.5) and set
\[
p := \lambda + \mu, \ q := |\mu| - |\lambda|, \quad (2.7)
\]
\[
\psi(p, q) := 16\pi^2 \tilde{f}(\lambda, \mu) |\lambda| |\mu| \exp(a(|\lambda| + |\mu|))
\] (2.8)
and the right side of (2.8) should be expressed as a function of \( (p, q) \) by formulas (2.7).

If \( \mu > 0 \) and \( \lambda > 0 \) then the point \((p, q)\) defined by (2.7) runs through \( Q_+ = \{(p, q): |q| < p, p > 0\} \).

If \( \lambda < 0 \) and \( \mu < 0 \) then \((p, q)\) runs through \( Q_- = \{(p, q): |q| < -p, p < 0\} \). If \( \psi(p, q) \) is known in \( Q_+ \) or \( Q_- \) then \( v(z) \) can be uniquely recovered from (2.6) by the analytical methods given in [2] p. 270-274, where inversion of the Fourier and Laplace transforms of compactly supported functions from a compact set is given. This inversion problem is ill-posed and its numerical implementation is not a simple matter.

One can use the same ideas to solve equation (2.4) at a fixed \( k > 0 \) in the Born approximation. The basic equation analogous to (2.4) for the case when \(-k < \lambda, \mu < k\), is:
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\[ \int v(z) \exp\{-i(pz^3+qz)\}dz = f(p,q) \text{ for } -k < \mu, \lambda < k \]  \hspace{1cm} (2.9)

where \( p = \lambda + \mu, q_1 := r(\mu) - r(\lambda), \)

\[ F(p,q_1) := -16\pi^2 i(\lambda,\mu) r(\lambda) r(\mu) \exp\{-i(\lambda + \mu)\} \] \hspace{1cm} (2.10)

and the right side of (2.10) should be expressed as a function of \( p, q_1. \)

If \((\lambda, \mu) \{ |\lambda| > k \text{ and } |\mu| > k, \lambda, \mu \text{ are real} \} \) then the basic equation in the Born approximation is equation (2.6) in which the right side is now given by the formula \( \hat{\psi} = F, \) where \( F \) is defined by (2.10) and in (2.10) the radicals \( r(\lambda) \) and \( r(\mu) \) are computed by formula (2.3) for \( \lambda^2 > k^2 \) and \( \mu^2 > k^2. \)

Equation (2.9) can also be solved analytically with the prescribed accuracy by the methods given in [2].

The problem considered is of interest in application.

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