ON SOME PROPERTIES OF POLYNOMIAL RINGS

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ABSTRACT. For a commutative ring with unity R , it is proved that R is a PF - ring if and only if the annihilator, $\text{ann}_R(a)$, for each $a \in R$ is a pure ideal in R . Also it is proved that the polynomial ring, $R[X]$, is a PF-ring if and only if R is a PF-ring. Finally, we prove that R is a PP-ring if and only if $R[X]$ is a PP-ring.

KEY WORDS AND PHRASES. Polynomial Rings, Pure ideal, PF-ring, PP-ring, R -flatness, and idempotent elements.

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1. INTRODUCTION.

All our rings in this paper are commutative with unity. An ideal I of a ring R is called pure if for any $x \in I$, there exists $y \in I$ such that $xy = x$. A ring is called a PF-ring if every principal ideal aR is a flat R -module. A ring R is called a PP-ring if every principal ideal aR is a projective R -module. One can easily show that aR is projective if and only if the annihilator, $\text{ann}_R(a)$, is generated by an idempotent element, (see [1], [2]).

First, we state a proposition characterizing flat R -modules elementwise. This is a well known result in commutative ring theory, (see [3]).

PROPOSITION 1. An R -module M is a flat R -module if and only if for any pair of finite subsets $\{x_1, x_2, \dots, x_n\}$ and $\{a_1, a_2, \dots, a_n\}$ of M and R respectively, such that

$$\sum_{i=1}^n x_i a_i = 0 \quad \text{there exists elements } z_1, \dots, z_k \in M \text{ and } b_{ij} \in R; i = 1, 2, \dots, k$$

$$\text{such that } \sum_{i=1}^n b_{ji} a_i = 0, j=1, 2, \dots, k, \text{ and } x_i = \sum_{j=1}^k z_j b_{ji}, i = 1, 2, \dots, n.$$

In the following theorem we establish that R is a PF-ring if and only if $\text{ann}_R(a)$ for each $a \in R$ is a pure ideal.

THEOREM 1. For any ring R , R is a PF-ring if and only if $\text{ann}_R(m)$ for each $m \in R$ is a pure ideal.

PROOF. Let $x_1, x_2, \dots, x_n \in mR$ and $a_1, a_2, \dots, a_n \in R$ with $\sum_{i=1}^n x_i a_i = 0$. Then there exists $m_1, m_2, \dots, m_n \in R$ such that $x_i = m_i m$, $i = 1, 2, \dots, n$. So

$$\sum_{i=1}^n m m_i a_i = 0. \quad \text{Hence } m \in \text{ann}_R\left(\sum_{i=1}^n m_i a_i\right).$$

Since $\text{ann}_R(\sum_{i=1}^n m_i a_i)$ is a pure ideal, there exists $b \in \text{ann}_R(\sum_{i=1}^n m_i a_i)$ such that $bm = m$.

Now take $m \in mR$ and $bm_1, bm_2, \dots, bm_n \in R$. These elements satisfy $\sum_{i=1}^n bm_i a_i = 0$ and $bm_i m = m_i m = x_i$, $i = 1, 2, \dots, n$. Therefore mR is a flat R -module.

Conversely, let $b \in \text{ann}_R(m)$. Then $mb = 0$. Since bR is a flat R -module, there exists $c \in bR$ and $d \in R$ such that $dm = 0$ and $b = cd$. Now $c = c_1 b$, so $b = cd = bc_1 d$. Moreover $c_1 d \in \text{ann}_R(m)$. Therefore $\text{ann}_R(m)$ is a pure ideal.

LEMMA 1. Let $I_1, I_2, \dots, I_n$ be a finite set of pure ideals of a ring R , then $J = \bigcap_{j=1}^n I_j$ is a pure ideal.

PROOF. Let $x \in J$. Then $x \in I_j$ for each j . Thus there exists $y_1 \in I_1, y_2 \in I_2, \dots, y_n \in I_n$ with $xy_j = x$, $j = 1, 2, \dots, n$. Then $y = y_1 y_2 \dots y_n \in J$ and $xy = x$.

Let R be a reduced (without nonzero nilpotent elements) ring. Let $h(X) = h_0 + h_1 X + \dots + h_n X^n \in R[X]$. Then $\text{ann}_{R[X]}(h(X)) = N[X]$, where N is the annihilator of the ideal generated by $h_0, h_1, \dots, h_n$, that is $N = \text{ann}_R(h_0, h_1, \dots, h_n) = \bigcap_{i=0}^n \text{ann}_R(h_i)$. Moreover if $f(X) = a_0 + a_1 X + \dots + a_m X^m \in \text{ann}_{R[X]}(h(X))$ then $a_i h_j = 0$ for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$ (see [4]).

LEMMA 2. Let R be a PF-ring, then R is reduced.

PROOF. Let a be a nilpotent element in R , $a \neq 0$. Let n be the least positive integer greater than 1 such that $a^n = 0$. Hence $a \in \text{ann}_R(a^{n-1})$. Since $\text{ann}_R(a^{n-1})$ is pure, there exists $b \in \text{ann}_R(a^{n-1})$ with $ab = a$. Now $0 = ba^{n-1} = a^{n-1}$ since $ba = a$.

Contradiction. Thus R is reduced.

THEOREM 2. The ring of polynomials, $R[X]$, is a PF-ring if and only if R is a PF-ring.

PROOF. Let $f(X) = a_0 + a_1 X + \dots + a_m X^m \in \text{ann}_{R[X]}(h(X))$ where $h(X) = h_0 + h_1 X + \dots + h_n X^n$.

Since $R[X]$ has no nonzero nilpotent elements,

$$a_i \in J = \bigcap_{j=0}^n \text{ann}_R(h_j), \quad i = 0, 1, 2, \dots, m.$$

By Lemma 1, J is pure. Hence there exist $b_1, b_2, \dots, b_m \in J$ such that $a_i b_i = a_i$,

$i = 1, 2, \dots, m$. Now our aim is to find $c \in J$ such that $c f(X) = f(X)$. We construct this element inductively.

First, $a_0 b_0 = a_0$. Consider

$$\begin{aligned} & (a_0 + a_1 X)(b_0 + b_1 - b_1 b_0) \\ &= a_0 b_0 + a_0 b_1 - a_0 b_0 b_1 + a_1 b_0 X + a_1 b_1 X - a_1 b_0 b_1 X \\ &= a_0 + a_0 b_1 - a_0 b_1 + a_1 b_0 X + a_1 X - a_1 b_0 X \\ &= a_0 + a_1 X. \end{aligned}$$

Let $c_1 = b_0 + b_1 - b_1 b_0$, then

$$\begin{aligned} & (a_0 + a_1 X + a_2 X^2)(c_1 + b_2 - c_1 b_2) \\ &= (a_0 + a_1 X)c_1 + b_2(1 - c_1)(a_0 + a_1 X) + a_2 c_1 X^2 + a_2 b_2 X^2 - a_2 b_2 c_1 X^2 \end{aligned}$$

$$\begin{aligned}
&= a_0 + a_1X + a_2c_1X^2 + a_2b_2X^2 - a_2c_1X^2 \\
&= a_0 + a_1X + a_2X^2
\end{aligned}$$

Similarly, $c_2 = c_1 + b_2 - c_1b_2, \dots$

$$\begin{aligned}
c_m &= c_{m-1} + b_m - c_{m-1}b_m \quad \text{and} \\
(a_0 + a_1X + \dots + a_iX^i) \quad c_i &= a_0 + a_1X + \dots + a_iX^i \\
i &= 0, 1, 2, \dots, m. \quad \text{Moreover } c_0, c_1, \dots, c_m \in J.
\end{aligned}$$

Thus there exist $c = c_m \in J$ with $cf(X) = f(X)$.

Conversely, assume $R[X]$ is a PF-ring. Let $a \in R$ and $b \in \text{ann}_R(a)$.

Then $b \in \text{ann}_{R[X]}(a)$. Since R is a PF-ring there exists

$$g(X) = c_0 + c_1X + \dots + c_kX^k \in \text{ann}_{R[X]}(a)$$

with $b g(X) = 0$. Hence $bc_0 = 0$ and $c_0a = 0$.

Consequently, R is a PF-ring.

THEOREM 3. R is a PP-ring if and only if $R[X]$ is a PP-ring.

PROOF. It is enough to show that $\text{ann}_{R[X]}(f(X))$ is generated by an idempotent element in $R[X]$, where $f(X) = a_0 + a_1X + \dots + a_nX^n$. Since R is reduced, $\text{ann}_{R[X]}(f(X)) = N[X]$ where N is the annihilator of the ideal generated by $a_0, a_1, \dots, a_n$.

$$\begin{aligned}
N &= \text{ann}_R(a_0, a_1, \dots, a_n) \\
&= \bigcap_{i=0}^n \text{ann}_R(a_i) \\
&= \bigcap_{i=0}^{\infty} e_i R, \quad e_i^2 = e_i \quad \text{because } R \text{ is a PP-ring.} \\
&= (e_1 e_2 \dots e_n) R \\
&= eR, \quad \text{where } e = e_1 e_2 \dots e_n
\end{aligned}$$

Hence $\text{ann}_{R[X]}(f(X)) = eR[X]$, $e^2 = e$

Conversely, let $R[X]$ be a PP-ring, let $a \in R$, then consider $\text{ann}_R(a)$. Since $R[X]$ is a PP-ring, $\text{ann}_{R[X]}(a) = g(X)R[X]$, where $g(X)^2 = g(X)$. If $g(X) = b_0 + b_1X + \dots + b_mX^m$, then $b_0^2 = b_0$. We claim $\text{ann}_R(a) = b_0R$. Let $b \in \text{ann}_R(a)$, then $ba = 0$. So $b \in g(X)R[X]$. Thus $b = (b_0 + b_1X + \dots + b_mX^m)(c_0 + c_1X + \dots + c_tX^t)$. Therefore $b = b_0c_0$, that is $b \in b_0R$.

For the other way around, let $b \in b_0R$. Then $b = b_0c_0$ for some $c_0 \in R$. Since $b_0a = 0$. That is $b \in \text{ann}_R(a)$. Thus $\text{ann}_R(a) = b_0R$.

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