FINITE $p'$-NILPOTENT GROUPS. II

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ABSTRACT: In this paper we continue the study of finite $p'$-nilpotent groups that was started in the first part of this paper. Here we give a complete characterization of all finite groups that are not $p'$-nilpotent but all of whose proper subgroups are $p'$-nilpotent.

KEY WORDS AND PHRASES. Frattini subgroup, $p'$-nilpotent group, maximal subgroup, nilpotent group, solvable group

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1. INTRODUCTION.
We consider only finite groups. The concept of $p'$-nilpotency was introduced in [1]. Briefly, a p-closed group is $p'$-nilpotent if it has a nilpotent Sylow p-complement. In this paper we consider groups which possess a large number of $p'$-nilpotent groups where the prime p remains the same for the several subgroups or it differs from subgroup to subgroup. Here we rely heavily on the theorem of N.Ito in which he proves that a minimal non-$p$-nilpotent group is a minimal non-nilpotent group. K.Iwasawa separately.

We show that a group in which every two generator proper subgroup is $p'$-nilpotent is either $p'$-nilpotent or a $p$-nilpotent minimal non-nilpotent group. Then we study the case when the proper subgroups are either $p'$-nilpotent or $q'$-nilpotent and show that such groups are always solvable. The main theorem of this paper completely classifies all simple groups with every proper subgroup $p'$-nilpotent for some prime p.

Notation and terminology are standard as in [2].

2. DEFINITIONS AND KNOWN RESULTS.
For the sake of completeness we give the following definition and result from [1].
DEFINITION 2.1: $G$ is a $\pi$-nilpotent group, $\pi$ a set of primes, if $G_\pi \lhd G$ and $G/G_\pi$, a nilpotent $\pi$-group. Let $P$ denote the set of all primes. When $\pi = P - \{p\}$, we say that $G$ is a $p'$-nilpotent group.
LEMMA 2.2: G is p'-nilpotent if and only if G is q-nilpotent ∀ q ≠ p. (see Corollary 2.4 of [1])

THEOREM 2.3: Let G be a group such that all proper subgroups are p-nilpotent but G is not p-nilpotent. Then

(i) every proper subgroup of G is nilpotent,
(ii) |G| = p^{aq_b}, p ≠ q,
(iii) G has a normal Sylow p-subgroup; for p > 2 exp (G_p) = p and for p = 2 the exponent is at most 4,
(iv) Sylow q-subgroups are cyclic. (see Satz 5.4 of [2])

Combining Lemma 2.2 and Theorem 2.3 we have the following theorem.

THEOREM 2.4: Let G be a group with the property that all its proper subgroups are p'-nilpotent for the prime p. Then G is either p'-nilpotent or G is a p-nilpotent minimal non-nilpotent group.

3. MINIMAL NON-p'-NILPOTENT GROUPS.

In Theorem 2.4 we required that all proper subgroups be p'-nilpotent. We now weaken the hypothesis in Theorem 2.4 by requiring only that those proper subgroups that are generated by two elements be p'-nilpotent.

THEOREM 3.1: Let G be a group with every proper subgroup generated by two elements p'-nilpotent for the prime p. Then G is either p'-nilpotent or G is a p-nilpotent SRI-group.

PROOF: Suppose G is not p'-nilpotent. Using 2.2 G is not q-nilpotent for some q ≠ p. Using Theorem 14.4.7, p217 of [3], there exists an r-element x and a q-subgroup Q such that x ∉ N_G(Q) - C_G(Q), r ≠ q. Consider H = Q<x>. Clearly |H| = q^{aq_b}.

CASE 1. r = p.

If H < G, then ∀ y ∈ Q, <x, y> is p'-nilpotent by hypothesis, i.e., <x, y> is p-closed. Since |H_p| = |x|, this means that y ∈ N_G(<x>) ∀ y ∈ Q; i.e., Q ≤ N_G(<x>) = G, i.e., H = Q<x>, a nilpotent group, i.e., x ∈ C_G(Q), a contradiction. Hence H = G with H_q = G_q = G and G_p = <x> ≠ G. Let K < G. Then K = Q_1<x^1> where Q_1 ≤ Q. G_q ⊲ G implies K_q ⊲ K. If K is generated by two elements, then K is p'-nilpotent by hypothesis, so K_p ⊲ K. Thus K is nilpotent. If K is not generated by two elements, then ∀ k ∈ K, <k, x^1> is p'-nilpotent and hence <k, x^1> is nilpotent. Hence x^1 ∈ C_G(k).

Thus x^1 commutes with all q-elements in K and hence K is nilpotent. Thus all proper subgroups of G are nilpotent, so G is a p-nilpotent minimal non-nilpotent group.

CASE 2. r ≠ p.

|H| = q^{aq_b}. Suppose H < G. ∀ y ∈ Q, <x, y> ≤ H < G. By hypothesis <x, y> is p'-nilpotent. p | |H| implies then that <x, y> is nilpotent. i.e., xy = yx ∀ y ∈ Q; i.e., x ∈ C_G(Q), a contradiction. Hence H = G. As in Case 1 we can conclude again that G is a p-nilpotent minimal non-nilpotent group. Q.E.D.
Since $p'$-nilpotency is inherited by subgroups the condition of 2.4 follows if all maximal subgroups of $G$ are $p'$-nilpotent. In 3.1 we required only the proper subgroups generated by two elements to be $p'$-nilpotent. In both cases $G$ was solvable. We now show that if we require only the core-free maximal subgroups to be $p'$-nilpotent, then $G$ is solvable under suitable conditions.

**Theorem 3.2:** Let $G$ be a group with at least one core-free maximal subgroup. If $G$ has the following properties:

(i) Sylow 2-subgroups of $G$ have all their proper subgroups abelian,

(ii) all core-free maximal subgroups of $G$ are $p'$-nilpotent for the prime $p$, then $G$ is solvable.

**Proof:** Suppose that all maximal subgroups of $G$ are core-free. By hypothesis then all maximal subgroups of $G$ are $p'$-nilpotent. Using 2.4 $G$ is then solvable. So assume that $G$ has at least one $M < G$ with $M_p \neq 1$. Thus $G$ is not a simple group. We now assume that $G$ is not solvable and arrive at a contradiction. First we show that all core-free maximal subgroups of $G$ are conjugate; clearly we can assume that $G$ has at least two core-free maximal subgroups $M_1$ and $M_2$. Let $N$ be a minimal normal subgroup of $G$. Then $G = M_1 N = M_2 N$, so $[G : N] = [M_1 : M_1 \cap N]$ and $[G : N] = [M_2 : M_2 \cap N]$.

**Case 1.** $p | [G : N]$.

Hence $p | |M_1|$, $i = 1, 2$. $M_i$ $p'$-nilpotent implies $M_i = N_G(P_i)$, where $P_i$ is the Sylow $p$-subgroup of $M_i$. Hence $P_i$ is a Sylow $p$-subgroup of $G$. Since $P_1$ and $P_2$ are conjugate, this means that $M_1$ and $M_2$ are conjugate.

**Case 2.** $p \nmid [G : N]$.

Hence $p \nmid |M_1|$ and $M_1$ are nilpotent. Just as in Case 1, $M_1$ will then be conjugate to $M_2$. Thus we assume that $p | |M_1|$ and $p \nmid |M_2|$. Hence $M_1 = N_G(P_1)$ and $M_2$ is nilpotent. Moreover, the argument of Case 1 shows that $M_2$ is a Hall subgroup of $G$. If $M_2$ is of odd order, then using Thompson's theorem on solvability of a group with a nilpotent maximal subgroup of odd order we see that $G$ is solvable. Since we have assumed that $G$ is not solvable, this means that $M_2$ is of even order. If $M_2$ is not a Sylow 2-subgroup of $G$, then using Satz 7.3, p.444 of [2] we see that $G = M_2 N$ with $M_2 \cap N = 1$. Since $2 \nmid |N|$, $N$ is solvable. Thus $N$ and $G/N$ are solvable implies $G$ is solvable. Hence we have by choice of $G$ that $M_2$ is a Sylow 2-subgroup of $G$. Hence $G = M_2 N$. Let $T$ be a Sylow 2-subgroup of $N$. Since $N \lhd G$ and $[G : N] = 2^n$, $N$ contains all Sylow $p$-subgroups of $G$ for $p \neq 2$. Hence $M_2 \cap N < M_2$. By hypothesis (i) $M_2 \cap N$ is abelian. $G/N$ is a 2-group. Now using
Satz 7.4, p.445 of [2] we get $M_2 \cap N = 1$. This is contrary to $M_2 \cap N \neq 1$. Thus, using previous arguments we see that $M_1$ and $M_2$ are conjugate. Suppose $G$ has another miniminormal subgroup $N_1 \neq N$. Then $G = M_1 N = M_1 N_1$. By hypothesis $M_1$ is $p'$-nilpotent, so $M_1$ is solvable. Hence $G = G/(N \cap N_1) \cong (G/N) \times (G/N_1)$ shows that $G$ is solvable.

By choice of $G$ this means that $G$ has a unique minimal normal subgroup of $G$. Since all core-free maximal subgroups of $G$ are conjugate they all have the same index in $G$. Now using Lemma 3, p.121 of [4] $N$ is solvable and hence $G$ is solvable. This final contradiction completes the proof. Q.E.D.

**COROLLARY 3.3**: Let $G$ be a group with the property that all of its nonnormal maximal subgroups are $p'$-nilpotent. If Sylow 2-subgroups of $G$ have all their proper subgroups abelian, then $G$ is solvable.

**PROOF**: Suppose that all maximal subgroups of $G$ are normal in $G$. Then $G$ is nilpotent and hence $G$ is solvable. On the other hand if $G$ has no normal maximal subgroups, then by hypothesis all maximal subgroups are $p'$-nilpotent and hence $G$ is solvable using 2.4. Assume now that $G$ has at least two nonnormal maximal subgroups $M_1, M_2$. By hypothesis $M_1, M_2$ are $p'$-nilpotent, hence solvable. Suppose that $M_2 \neq 1$. If $M_2 \not\subseteq M_1$, then $G = M_2 M_1$. $M_2$ and $G/M_2$ are solvable implies that $G$ is solvable. Assume that $M_2 \subseteq M_1$. Hence $M_2 \leq (M_1)_G$. Using a similar argument with $(M_1)_G$ we have $(M_1)_G \leq M_2$. Hence $M_2 = (M_1)_G$; i.e., all nonnormal maximal subgroups having nontrivial core have the same core. If all nonnormal maximal subgroups have nontrivial core, then by the above argument they have the same core, say $N$. Consider $G/N$. Using 3.2 $G/N$ is solvable and since $N$ is solvable we have $G$ solvable. Finally, if all the nonnormal maximal subgroups are core-free, then using 3.2 $G$ is solvable. Q.E.D.

So far we considered the condition that many subgroups of $G$ are $p'$-nilpotent for the same prime $p$. In the next theorem we consider the situation that the proper subgroups are either $p'$-nilpotent or $q'$-nilpotent.

**THEOREM 3.4**: Let $G$ be a group with the property that all its proper subgroups are either $p'$-nilpotent or $q'$-nilpotent, $p \neq q$ are primes that are fixed. Then $G$ is solvable.

**PROOF**: If $G$ is $p'$-nilpotent or $q'$-nilpotent, then $G$ is solvable. Assume that $G$ is neither $p'$-nilpotent nor $q'$-nilpotent. If $|G|$ is divisible by $p$ and $q$ alone, then using Burnside's theorem on solvability of groups of order $p^aq^b$, $G$ is solvable. Assume that $|G|$ has at least 3 distinct primes, say $p, q, r$. By hypothesis all proper subgroups of $G$ are $r$-nilpotent using Lemma 2.2. Using Theorem 2.3 we see that $G$ is $r$-nilpotent; i.e. $G^r \triangleleft G$ and $G = G_r G^r$ where $G^r$ is the Sylow $r$-complement of $G$. $G^r$ is solvable by hypothesis and $G/G^r = G_r$ is solvable. Hence $G$ is solvable. Q.E.D.
EXAMPLE 3.5: Let $G = A_5$. Every proper subgroup of $G$ is either $2'$-nilpotent, $3'$-nilpotent or $5'$-nilpotent. $G$ is not solvable.

This example shows that in Theorem 3.4 we can not, in general, replace 2 primes by 3 primes.

4. MAIN THEOREM.

Example 3.5 shows that when we vary the prime $p$ in the requirement that all proper subgroups be $p'$-nilpotent, then the group need not be solvable. In this section we completely classify all finite simple groups with this property. First we prove the following lemma.

**Lemma 4.1:** Let $G$ be nonnilpotent dihedral group of order $2m$. If $G$ is $p'$-nilpotent, then $m = 2^ap^b$.

Next we state and prove the main theorem. In the proof of this theorem we will need Thompson's classification of minimal simple groups and Dickson's list of all subgroups of $\text{PSL}(2, p^n)$. Also, we need details of the Suzuki group which are given in [5].

**Main Theorem:** Let $G$ be a nonsolvable simple group with the property that all its proper subgroups are $q'$-nilpotent for some arbitrary prime $q$. Then $G$ is one of the following types:

(a) $\text{PSL}(2, p)$, with $p^2 - 1 \not\equiv 0 \pmod{5}$, $p^2 - 1 \not\equiv 0 \pmod{16}$, $p > 3$, $p - 1 = 2^ir^i$ and $p + 1 = 2s^j$ or $p - 1 = 2r^i$ and $p + 1 = 2^2s^j$ where $r,s$ are odd primes, $i,j \geq 0$.

(b) $\text{PSL}(2, 2^n)$, $n$ is a prime, $2^n - 1 = r^i$, $2^n + 1 = s^j$, $r,s,i,j$ as in (a).

(c) $\text{PSL}(2, 3^n)$, $n$ is an odd prime, $3^n - 1 = 2^ir^i$ and $3^n + 1 = 2s^j$ or $3^n - 1 = 2r^i$ and $3^n + 1 = 2^2s^j$, $r,s,i,j$ as in (a).

Conversely, if $G$ is one of the groups listed above in (a), (b) or (c), then $G$ is a simple group with all its proper subgroups $q'$-nilpotent for some prime $q$.

**Proof:** Since a $q'$-nilpotent group is always solvable, all proper subgroups of $G$ are solvable. Hence using Thompson's list of minimal simple groups (see [6]), we conclude that $G$ is one of the following types:

(i) $\text{PSL}(2, p)$ where $p > 3$, $p^2 - 1 \not\equiv 0 \pmod{5}$,

(ii) $\text{PSL}(2, 2^r)$, $r$ is a prime,

(iii) $\text{PSL}(2, 3^r)$, $r$ is an odd prime,

(iv) $\text{PSL}(3, 3)$,

(v) the Suzuki group $Sz(2^r)$ where $r$ is an odd prime.

Now we use the subgroups of $\text{PSL}(2, p^f)$ listed in Hauptsatz 8.27, pp. 213-214 of [2]. For easy reference we give this list below and refer to it as Dickson's list. Dickson's list of subgroups of $\text{PSL}(2, p^f)$:

(i) elementary abelian $p$-groups,

(ii) cyclic groups of order $z$ with $z|(p^f - 1)/k$, where $k = (p^f - 1, 2)$,
(iii) dihedral groups of order $2z$ where $z$ is as in (ii),
(iv) alternating group $A_4$ for $p \neq 2$ or $p = 2$ and $f \equiv 0 \text{mod } 2$,
(v) symmetric group $S_4$ for $p^{2f} - 1 \equiv 0 \text{mod } 16$,
(vi) alternating group $A_5$ for $p = 5$ or $p^{2f} - 1 \equiv 0 \text{mod } 5$,
(vii) semidirect product of elementary abelian group of order $p^m$ with cyclic group of order $t$ with $t | (p^m - 1)$ and $t | (p^f - 1)$,
(viii) groups $PSL(2, p^m)$ for $m | f$.

In Dickson's list, the subgroups in (i), (ii), (iv) and (vii) are $q'$-nilpotent for some prime $q$. Using the possible choices for $G$ listed above, Dickson's list (viii) can not be a subgroup of $G$. $S_4$ is not $q'$-nilpotent for any prime $q$. Hence using Dickson's list (v) we have $p^{2f} - 1 \not\equiv 0 \text{mod } 16$. Also, $A_5$ being a simple group can not be a proper subgroup of $G$. Thus, from Dickson's list (vi) we have $p^{2f} - 1 \not\equiv 0 \text{mod } 5$.

Using Lemma 4.1, $z = 2^a v^b$ where $v$ is a prime. Using these observations and Lemma 4.1 it is a matter of routine verification that the Thompson's list of groups (i) - (iii) given earlier would be a choice for $G$.

(i) $PSL(3, 3)$.

Considering $K = PSL(3, 3)$ as a doubly transitive group on 13 letters, the stabilizer of a point will be a maximal subgroup $M$ with $|M| = 3^3 \cdot 2^4$. $M \cong GL(2, 3) \cdot (Z_3 \times Z_3)$ shows that $M$ is not $p'$-nilpotent for any prime $p$. So $PSL(3, 3)$ can not be a choice for $G$.

(ii) $Sz(2^q)$, $p$ an odd prime.

Using the notation and results used in Suzuki [5], we will now verify that $Sz(2^q)$ has a subgroup, namely $N_L(A_1)$, which is not $s'$-nilpotent for any prime $s$, and thus $Sz(2^q)$ can not be a choice for $G$.

CASE 1: $s = 2$.

Using Proposition 15, p.121 of [5], $N_L(A_1)/A_1$ is cyclic. If $N_L(A_1)$ is $2'$-nilpotent, since $|N_L(A_1)/A_1| = 4$ and $|A_1|$ is an odd number, we will have $N_L(A_1)$ to be nilpotent. Hence every element of odd order commutes with every 2-element. This is contrary to Lemma 11, p.135 of [5]. Hence $N_L(A_1)$ can not be $2'$-nilpotent.

CASE 2: $s \neq 2$.

In this case $N_L(A_1)$ has an abelian subgroup which is a complement of a Sylow $s$-subgroup of $N_L(A_1)$. Again, using Lemma 11, p.135 of [5], such a subgroup does not exist. Thus $N_L(A_1)$ is not $s'$-nilpotent for any prime $s$. Thus $Sz(2^q)$ can not be a choice for $G$.

Conversely, suppose that $G$ is one of the groups listed in the statement. Clearly all the groups are simple. First consider $G = PSL(2, p)$ as in (a). From the list of subgroups of $PSL(2, p)$ given in Dickson's list, the subgroups in (i), (ii), (iv) and (vii) are $q'$-nilpotent for some prime $q$. (v) and (vi) can not be subgroups of $G$ because $p^2 - 1 \not\equiv 0 \text{mod } 5$ and $p^2 - 1 \not\equiv 0 \text{mod } 16$.
Suppose $G$ has a subgroup $H$ as in (iii). $|G| = p(p^2 - 1)/2$, $|H| = 2z$ with $z \mid (p \pm 1)/2$. Suppose $z \mid (p - 1)/2$, $(p - 1)/2 = 2^2r^1/2 = 2r^1$. $z \mid 2r^1$. $|H| = 2z$. Hence $H$ has a cyclic normal subgroup of order $z$, say $K$. If $|K| = r^1$ where $1 \leq i$, then $|H| = 2r^1$ and hence $H$ is $r'$-nilpotent. If $|K| = 2r^1$, then $K_r$ char $K$ $H$ implies $K_r$. $H$. Also, $K_r = H_r$ since $|H| = 2^2r^1$. Thus $H$ is $r'$-nilpotent in this case as well.

Suppose $z \mid (p + 1)/2$. If $p + 1 = 2^s$ $j$, then as in the above argument we get $H$ to be $q'$-nilpotent for some prime $q$, so assume that $p + 1 = 2^s$ $j$. $z \mid (p + 1)/2 = 2s^j/2 = s^j$. Thus $z = s^j$ where $1 \leq j$. Clearly $H$ is $s'$-nilpotent in this case as noted in the previous argument. Thus all proper subgroups of $G$ are $q'$-nilpotent for some prime $q$ when $G$ is as in (a).

Next consider $G = PSL(2, 2^n)$ as in (b). In this case $z \mid (2^n + 1)$ and $2^n - 1 = r^1$, $2^n + 1 = s^1$ where $r, s$ are odd primes. Thus if $H$ is a subgroup of $G$ of order $2z$, then clearly $H$ is $q'$-nilpotent for some prime $q$. Thus all proper subgroups of $G = PSL(2, 2^n)$ as in (b), are $q'$-nilpotent for some prime $q$. Finally consider $G = PSL(2, 3^n)$ as in (c). In this case $z \mid (3^n + 1)/2$. The argument given earlier for the case $G = PSL(2, p)$ applies here as well. Thus we complete the proof of the main theorem. Q.E.D.

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