ON STAR POLYNOMIALS, GRAPHICAL PARTITIONS AND RECONSTRUCTION

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ABSTRACT. It is shown that the partition of a graph can be determined from its star polynomial and an algorithm is given for doing so. It is subsequently shown (as it is well known) that the partition of a graph is reconstructible from the set of node-deleted subgraphs.

KEY WORDS AND PHRASES. *star, star polynomial, star cover, graphical partition, reconstruction.*

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1. INTRODUCTION.

The graphs here will be finite, undirected, and will have no loops or multiple edges. We define an *m*-star S_m to be a tree consisting of a node of valency m (called the centre of S_m) joined to m other nodes. A *0*-star is a node and a *1*-star is an edge.

Let G be a graph. A *star-cover* (or simply, a *cover*) of G is a spanning sub-graph whose components are all stars. Let us associate with each m -star in G , an indeterminate or *weight* w_{m+1} ; and with each star cover C consisting of $S_{m_1}, S_{m_2}, \dots, S_{m_k}$; the weight

$$w(C) = \prod_{i=1}^k w_{m_i}.$$

Then the *star polynomial* of G (relative to the given weight assignment) is

$$E(G; \underline{w}) = \sum w(C),$$

where the summation is taken over all the star covers in G and $\underline{w} = (w_1, w_2, \dots)$ is a general weight vector. The basic results on star polynomials are given in the introductory paper by Farrell [1].

We will show that Π_G - the partition of a graph G can be obtained from $E(G; \underline{w})$. This will then be used to show that Π_G is node-reconstructible, a result that can be established by more elementary means (see Tutte [2]).

For brevity, we will write $E(G)$ for $E(G; \underline{w})$, since the same weight vector $\underline{w}$ will be used throughout the paper. Also, in partitions, we will use r^k to denote the occurrence of k r 's. Finally, we will assume that $\binom{n}{r} = 0$, for all $r > n$.

2. STAR POLYNOMIALS AND GRAPHICAL PARTITIONS.

First of all, we state a lemma which can be easily proved.

LEMMA 1. Let v be a node of valency d in G . Then G contains $\binom{d}{m}$ m -stars with centre v .

DEFINITION. Let G be a graph with p nodes. A *simple m -cover* of G is a cover consisting of an m -star and $p-m-1$ isolated nodes.

It is clear that a simple m -cover in G will have weight $w_1^{p-m-1} w_{m+1}$ in $E(G)$. A monomial of this form in $E(G)$ will be referred to as a *simple term*, and its coefficient, which will be denoted by c_m - a *simple coefficient*. c_m will be the number of simple m -covers in G . Note that the term w_p will also be a simple term.

LEMMA 2. Let G be a graph with p nodes. Let $\Pi_G = (n^n, \dots, 2^{b_2}, 1^{b_1}, 0^{b_0})$ ($0 \leq n \leq p-1$). Then for $m > 1$,

$$c_m = \sum_{r=m}^n \binom{r}{m} b_r.$$

PROOF. This is straightforward.

The following lemma can be easily proved.

LEMMA 3. Let n be the largest valency of a node in G . Then $E(G)$ contains all the terms $w_1^{p-r-1} w_{r+1}$ ($0 \leq r \leq n$) with non-zero coefficient. i.e. $c_r \neq 0$ for $0 \leq r \leq n$.

Suppose that we put $n = 1$ in the above Lemma. Then G will consist of a set of disjoint edges and possibly isolated nodes. Clearly then the partition of G will be given by

$$\Pi_G = (1^{c_1}, 0^{p-2c_1}). \quad (2.1)$$

Hence Π_G can be found from $E(G)$.

If $n > 1$, then from Lemma 2, we get

$$\begin{aligned} c_k &= \sum_{r=k}^n \binom{r}{k} b_r = b_k + \sum_{r=k+1}^n \binom{r}{k} b_r, \quad \text{for } k > 1. \\ \Rightarrow b_k &= c_k - \sum_{r=k+1}^n \binom{r}{k} b_r. \end{aligned} \quad (2.2)$$

Therefore b_k can be obtained from c_k (which is defined in $E(G)$) and all the higher b 's i.e. $b_{k+1}, b_{k+2}, \dots, b_n$. For $k = 1$ and 0 , we can find b_1 and b_0 as follows: c_1 is the number of edges in G . Therefore the sum of the valencies of the

$$\begin{aligned} \text{nodes of } G \text{ is } 2c_1 &= \sum_{k=1}^n k b_k = b_1 + \sum_{k=2}^n k b_k. \\ b_1 &= 2c_1 - \sum_{k=2}^n k b_k. \end{aligned} \quad (2.3)$$

b_0 is the number of isolated nodes in G . Therefore

$$b_0 = p - \sum_{i=1}^n b_i. \quad (2.4)$$

Our discussion yields the following theorem.

THEOREM 1. Let G be a graph with p nodes and let $\Pi_G = (n^{b_n}, \dots, 2^{b_2}, 1^{b_1}, 0^{b_0})$, where $0 \leq n \leq p-1$. Then

$$b_k = c_k - \sum_{r=k+1}^n \binom{r}{k} b_r, \quad \text{for } 1 \leq k \leq n.$$

$$b_1 = 2c_1 - \sum_{r=2}^n r b_r$$

and

$$b_0 = p - \sum_{r=1}^n b_r.$$

For $n = 1$, Π_G is given by Equation (2.1). For $n = 0$, the result is trivial.

Theorem 1 yields an algorithm for obtaining Π_G from $E(G)$. This algorithm is illustrated in the following example:

EXAMPLE 1. Let G be a graph such that

$$E(G) = w_1^6 + 6w_1^4w_2 + 7w_1^3w_3 + w_1^2w_4 + 8w_1^2w_2^2 + 12w_1w_2w_3 + w_2^3 + w_2w_4 + 3w_3^2$$

First of all, we observe from the term w_1^6 , that G has 6 nodes i.e. $p = 6$. The simple terms in $E(G)$ are $6w_1^4w_2$, $7w_1^3w_3$ and $w_1^2w_4$. Therefore $c_1 = 6$, $c_2 = 7$ and $c_3 = 1$. Since the largest k for which w_k occurs in $E(G)$ is 4, it follows that $n = 3$.

Now

$$b_3 = c_3 = 1; \quad b_2 = c_2 - \sum_{r=3}^3 \binom{r}{2} b_r = 7 - 3b_3 = 7-3 = 4.$$

$$b_1 = 2(6) - \sum_{k=2}^3 k b_k = 12 - 2b_2 - 3b_3 = 12-8-3 = 1.$$

$$b_0 = p - \sum_{r=1}^3 b_r = 6-1-4-1 = 0.$$

Hence $\Pi_G = (3^1 2^4 1^1)$.

It would be nice to be able to obtain G itself from $E(G)$. From Theorem 1, Π_G can be obtained. However there can be several graphs with the same partition. Since only the simple terms in $E(G)$ are used to obtain Π_G , it is not surprising that G itself is not clearly defined. Should G itself be clearly defined by the simple terms, then it would mean that the remaining terms of $E(G)$ are useless as far as the characterization of G is concerned. It would be interesting to investigate the nature of these 'useless terms'.

Suppose that Π_G is unigraphic (i.e. there is only one graph with partition Π_G), then G could be uniquely constructed from Π_G . Hence we have the following theorem.

THEOREM 2. Let Π_G be unigraphic. Then G can be constructed from $E(G)$.

PROOF. This follows from Theorem 1 and the discussion above.

THEOREM 3. Let G and H be two graphs with p nodes. Then $\Pi_G = \Pi_H$ if and only if $E(G)$ and $E(H)$ have the same simple coefficients.

PROOF. Suppose that the simple coefficients in $E(G)$ and $E(H)$ are equal. Then from Theorem 1, G and H must have the same partition. Conversely, suppose that $\Pi_G = \Pi_H$. Then from Lemma 2, the coefficients c_r ($r > 1$) in $E(G)$ and $E(H)$ must be equal. Finally, $c_0 = 1$, for all graphs. Hence $E(G)$ and $E(H)$ have the same simple coefficients. The result therefore follows.

3. STAR POLYNOMIALS AND RECONSTRUCTION.

The following theorem is analogous to the result for circuit polynomials given in Lemma 3 of Farrell and Grell [3], with $i = 1$. We suspect that the general result holds for all F -polynomials (see Farrell [4]). Here $G-x$ denotes the graph obtained from G by removing node x . $V(G)$ is the node set of G .

THEOREM 4.

$$\frac{\partial E(G)}{\partial w_1} = \sum_{x \in V(G)} E(G-x; \underline{w})$$

PROOF. Let us write $E(G) = \sum_j A_j w_1^{n_{1,j}} w_2^{n_{2,j}} \dots w_p^{n_{p,j}}$, where p is the number of nodes in G . Then

$$\frac{\partial E(G)}{\partial w_1} = \sum_j n_{1,j} A_j w_1^{n_{1,j}-1} w_2^{n_{2,j}} \dots w_p^{n_{p,j}}. \quad (3.1)$$

It is clear that the monomial $w_1^{n_{1,j}-1} w_2^{n_{2,j}} \dots w_p^{n_{p,j}}$ is the weight of a cover with one isolated node less than the corresponding cover in G . It is therefore the weight of a cover in $G-x$, for some node x in G . Hence it is a monomial of the polynomial $\sum E(G-x; \underline{w})$. Conversely, every cover of $G-x$ can be extended to a cover of G by adding an isolated node. Therefore every monomial m in $\sum E(G-x; \underline{w})$ yields a corresponding monomial $w_1 m$ in G . The derivative of $w_1 m$ with respect to w_1 yields a term with monomial m . It follows that $\frac{\partial E(G)}{\partial w_1}$ and $\sum E(G-x; \underline{w})$ have the same monomials.

Since A_j is the coefficient of $w_1^{n_{1,j}} w_2^{n_{2,j}} \dots w_p^{n_{p,j}}$, G has A_j covers consisting of $n_{1,j}$ isolated nodes, $n_{2,j}$ edges, ..., $n_{p,j}$ $(p-1)$ -stars. Suppose that node x is removed from G . Then $G-x$ will have a similar cover but with $n_{1,j}-1$ isolated nodes. Since node x could be any of the $n_{1,j}$ isolated nodes in the cover, it follows that each such cover in G gives rise to $n_{1,j}$ covers with one less isolated node. Hence the coefficient of $w_1^{n_{1,j}-1} w_2^{n_{2,j}} \dots w_p^{n_{p,j}}$ in $\sum E(G-x; \underline{w})$ is $n_{1,j} A_j$. From Equation (3.1) it follows that the monomials occur in $\sum E(G-x; \underline{w})$ and $\frac{\partial E(G)}{\partial w_1}$ with equal coefficients. Hence the result follows.

Throughout the rest of this section, we will assume that the graph has at least three nodes. Also 'reconstructible' would mean node-reconstructible. By the *deck*

D_G a graph G we would mean the set $\{G-x: x \in V(G)\}$.

It is well known (see Harary [5], Kelly [6], and Chatrand and Kronk [7]) that disconnected graphs are reconstructible. It follows that Π_G is reconstructible if G is disconnected, and therefore the number of isolated nodes in G is reconstructible. We can however, prove the latter independently. The following lemma gives a connection between the number of isolated nodes and D_G .

LEMMA 4. Let G be a graph with p nodes. Then G has r (>0) isolated nodes if and only if D_G has exactly r graphs with $r-1$ isolated nodes and $p-r$ graphs with r or more isolated nodes.

PROOF. Suppose that G has r isolated nodes. Then D_G must be of the form described. Conversely, suppose that D_G is as described in the theorem. Let k be the number of isolated nodes in G . Since every element of D_G has at least $r-1$ isolated nodes, G cannot have less than $r-1$ isolated nodes or it would mean that the removal of each node from G yields at least one new isolated node, and this is impossible unless G is a matching (in which case the result holds). Therefore $k \geq r-1$. Since D_G contains no elements with less than $r-1$ isolated nodes, $k \neq r-1$. Hence $k > r-1$. But exactly r elements of D_G has $r-1$ isolated nodes and only one node can be removed at a time from G to form an element of D_G . Therefore G has exactly r nodes each of whose removal reduces k to $r-1$. These nodes must be themselves isolated nodes. $\Rightarrow k \geq r$. Clearly $k \neq r$. Therefore $k = r$ and the result follows.

From the above lemma, we see that b_0 (of Theorem 1) can be obtained from D_G . It follows that if D_G is given, then we can find Π_G - provided that all the remaining b_r 's ($0 < r \leq n$) can be determined. The following result is well known (see Tutte [2]). We will give different derivation using star polynomials.

THEOREM 5. Π_G is reconstructible.

PROOF. Let G be a graph with p nodes. From Theorem 4, we have, by integrating both sides with respect to w_1 ,

$$E(G) = \int (\sum E(G-x; \underline{w})) + C(w_2, w_3, \dots, w_p),$$

where $C(w_2, w_3, \dots, w_p)$ is a polynomial in the weights $w_2, w_3, \dots, w_p$.

Suppose that D_G is given. Then $\sum E(G-x; \underline{w})$ can be found. Hence $\int \sum E(G-x; \underline{w})$ can be found. But this polynomial contains all the simple terms except w_p . Therefore all the simple coefficients of $E(G)$, except c_{p-1} can be immediately found. We will consider two cases (i) $c_{p-2} = 0$ and (ii) $c_{p-2} \neq 0$.

If $c_{p-2} = 0$, then from Lemma 3, $c_{p-1} = 0$. Therefore all the simple coefficients will be defined. It follows from Theorem 1, that Π_G can be found. If $c_{p-2} \neq 0$ then c_{p-1} will be unknown. However b_0 will be known from D_G (Lemma 4).

Therefore the system of equations in Theorem 1 will have $p-1$ equations and $p-1$ unknowns. It can be solved to find $b_1, b_2, \dots, b_{p-1}$. Hence Π_G can be found.

In the proof of Theorem 5, we did not give any useful practical method for finding Π_G , when $c_{p-2} \neq 0$. We shall do so now.

The equations of Theorem 1 can be written as follows:

$$p = b_{p-1} + b_{p-2} + \dots + b_2 + b_1 + b_0$$

$$2c_1 = \binom{p-1}{1}b_{p-1} + \binom{p-2}{1}b_{p-2} + \dots + 2b_2 + b_1$$

$$c_2 = \binom{p-1}{2}b_{p-1} + \binom{p-2}{2}b_{p-2} + \dots + b_2$$

.

$$c_{p-2} = \binom{p-1}{p-2}b_{p-1} + \binom{p-2}{p-2}b_{p-2}.$$

By adding and subtracting alternate equations, we get

$$p-2c_1 + c_2 - c_3 \dots + (-1)^p c_{p-2} = b_0 + (-1)^p b_{p-1}. \quad (3.2)$$

Let us denote the L.H.S. of this equation by S . Then

$$S = b_0 + (-1)^p b_{p-1}.$$

$$\Rightarrow b_{p-1} = (-1)^p (S - b_0) \quad (3.3)$$

Since S can be found from the simple coefficients c_r ($r \geq p-2$) and p , and b_0 can be found (from Lemma 5), b_{p-2} can be found from Equation (3.3). Hence all the b 's can be found by using the algorithm suggested by Theorem 1.

The following example illustrates the technique.

EXAMPLE 2. Let G be a graph with D_G given below in Figure 1.

Figure 1.

It can be easily confirmed that

$$E(G_1) = w_1^4 + 6w_1^2w_2^2 + 12w_1w_3 + 3w_2^2 + 4w_4$$

$$E(G_2) = E(G_3) = w_1^4 + 4w_1^2w_2 + 5w_1w_3 + w_2^2 + w_4$$

$$E(G_4) = E(G_5) = w_1^4 + 5w_1^2w_2 + 8w_1w_3 + 2w_2^2 + 2w_4.$$

Therefore

$$\sum_{i=1}^5 E(G_i) = \frac{\partial E(G)}{\partial w_1} = 5w_1^4 + 24w_1^2w_2 + 38w_1w_3 + 9w_2^2 + 10w_4.$$

By integrating with respect to w_1 , we get

$$E(G) = w_1^5 + 8w_1^3w_2 + 19w_1^2w_3 + 10w_1w_4 + C(w_2, w_3, w_4, w_5).$$

Therefore

$$p = 5, c_1 = 8, c_2 = 19 \text{ and } c_3 = 10.$$

$$\Rightarrow S = 5 - 16 + 19 - 10 = -2.$$

Since no element of D_G has an isolated node, it follows from Lemma 5 that $b_0 = 0$.

From Equation (3.3), we get

$$b_4 = (-1)^5 (-2-0) = 2.$$

$$b_3 = c_3 - \sum_{r=4}^4 \binom{r}{3} b_r = c_3 - 4b_4 = 10 - 8 = 2.$$

$$b_2 = c_2 - \sum_{r=3}^4 \binom{r}{2} b_r = c_2 - \binom{3}{2} b_3 - \binom{4}{2} b_4 = 19 - 6 - 12 = 1.$$

$$b_1 = 2c_1 - \sum_{r=2}^4 \binom{r}{1} b_r = 2c_1 - 2b_2 - 3b_3 - 4b_4 = 16 - 2 - 6 - 8 = 0.$$

Hence we get

$$\Pi_G = (4^3, 3^2, 2^1).$$

Suppose that G is unigraphic. Then G is uniquely determined from Π_G . But Π_G is reconstructible from D_G (Theorem 5). It follows that G is uniquely reconstructible from D_G . Thus we have the following theorem, which can otherwise be proved by more elementary methods.

THEOREM 6. The Reconstruction Conjecture holds for all graphs with unigraphic partitions.

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