SOME CONGRUENCE PROPERTIES OF BINOMIAL COEFFICIENTS AND LINEAR SECOND ORDER RECURRENCES

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ABSTRACT. Using elementary methods, the following results are obtained: (I) If p is

prime, $0 \leq m \leq n$, $0 < b < ap^{n-m}$, and $p \nmid ab$, then $\binom{ap^n}{bp^m} \equiv (-1)^{p-1} \binom{ap^{n-m}}{b} \pmod{p^n}$;

If r, s are the roots of $x^2 = Ax - B$, where $(A, B) = 1$ and $D = A^2 - 4B > 0$, if

$u_n = \frac{r^n - s^n}{r - s}$, $v_n = r^n + s^n$, and $k > 0$, then (II) $v_{kp^n} \equiv v_{kp^{n-1}} \pmod{p^n}$;

(III) If p is odd and $p \nmid D$, then $u_{kp^n} \equiv \left(\frac{D}{p}\right) u_{kp^{n-1}} \pmod{p^n}$;

(IV) $u_{k2^n} \equiv (-1)^B u_{k2^{n-1}} \pmod{2^n}$.

KEYWORDS AND PHRASES. Binomial coefficient, linear second order recurrence.

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1. INTRODUCTION.

Following Lucas [1], let A, B be integers such that $(A, B) = 1$ and

$D = A^2 - 4B > 0$. Let the roots of $x^2 = Ax - B$ be: $r = \frac{1}{2}(A + D^{1/2})$, $s = \frac{1}{2}(A - D^{1/2})$.

Let $n > 0$. Let the sequences u_n, v_n be defined by:

$$u_n = \frac{r^n - s^n}{r - s} \tag{1.1}$$

$$v_n = r^n + s^n \tag{1.2}$$

then

$$u_0 = 0, u_1 = 1, u_n = Au_{n-1} - Bu_{n-2} \text{ for } n \geq 2 \tag{1.3}$$

$$v_0 = 2, v_1 = A, v_n = Av_{n-1} - Bv_{n-2} \text{ for } n \geq 2 \tag{1.4}$$

$$r + s = A \tag{1.5}$$

$$rs = B \quad (1.6)$$

$$r-s = D^{1/2} \quad (1.7)$$

$$u_{2n} = u_n v_n \quad (1.8)$$

$$v_{2n} = v_n^2 - 2B^n \quad (1.9)$$

$$2u_{m+n} = u_m v_n + u_n v_m \quad (1.10)$$

Let p be prime. Let $O_p(t) = j$ if $p^j \mid t$, $p^{j+1} \nmid t$.

$$\text{If } 0 < k < p, \text{ then } p \mid \binom{p}{k}. \quad (1.11)$$

$$\text{If } 0 < k < n, \text{ then } \binom{n}{n-k} = \binom{n}{k} \quad (1.12)$$

$$a^p \equiv a^{p^{n-1}} \pmod{p^n} \quad (1.13)$$

$$\text{If } x \equiv \pm a \pmod{2^n}, \text{ then } x^2 \equiv a^2 \pmod{2^{n+1}} \quad (1.14)$$

$$\text{If } 0 < k < p^e \text{ and } p \nmid k, \text{ then } O_p\left(\binom{p^e}{k}\right) = e \quad (1.15)$$

REMARK. (1.1) through (1.10) appear in Lucas [1]. (1.11) through (1.14) are elementary. (1.15) is theorem 2 in Robbins [2].

2. MAIN RESULTS.

Lemma (2.1), If $0 < m < n$, $0 < b < ap^{n-m}$, and $p \nmid ab$,

then $\binom{ap^n}{bp^m} \equiv \binom{ap^{n-m}}{b} (-1)^{b(p^m-1)} \pmod{p^n}$.

PROOF. $\binom{ap^n}{bp^m} = \prod_{j=0}^{bp^m-1} \frac{ap^{n-j}}{bp^{m-j}} = P_1 P_2$, where P_1 is the product of

all such factors where $p^m \mid j$, and P_2 is the product of all such factors where $p^m \nmid j$.

Now $P_1 = \prod_{i=0}^{b-1} \frac{ap^{n-i}p^m}{bp^{m-i}p^m} = \prod_{i=0}^{b-1} \frac{ap^{n-m-i}}{b-i} = \binom{ap^{n-m}}{b}$, while

$P_2 = \prod_{\substack{j=1 \\ j \nmid p^m}}^{bp^m-1} \frac{ap^{n-j}}{j} \equiv (-1)^{bp^m-b} \pmod{p^n}$. Therefore

$$\binom{ap^n}{bp^m} \equiv \binom{ap^{n-m}}{b} (-1)^{b(p^m-1)} \pmod{p^n}.$$

THEOREM 2.1. If $0 < m < n$, $0 < b < ap^{n-m}$, and $p \nmid ab$,

then $\binom{ap^n}{bp^m} \equiv (-1)^{p-1} \binom{ap^{n-m}}{b} \pmod{p^n}$.

PROOF. If p is odd, then $b(p^m-1) \equiv p-1 \equiv 0 \pmod{2}$; if $p = 2$, then by hypothesis, b is odd, so $b(2^m-1) \equiv 1 \equiv 2-1 \pmod{2}$. In either case, the conclusion now follows from Lemma 2.1.

LEMMA 2.2. $v_{2n+1} = A^{2n+1} - \sum_{k=1}^n \binom{2n+1}{k} B^k v_{2n+1-2k}$

PROOF. By (1.2) and the binomial theorem, we have

$$v_{2n+1} = r^{2n+1} + s^{2n+1} = (r+s)^{2n+1} - \sum_{k=1}^{2n} \binom{2n+1}{k} r^{2n+1-k} s^k, \text{ so (1.5) implies}$$

$$v_{2n+1} = A^{2n+1} - \left\{ \sum_{k=1}^n \binom{2n+1}{k} r^{2n+1-k} s^k + \sum_{k=1}^n \binom{2n+1}{2n+1-k} r^k s^{2n+1-k} \right\}. \text{ Now (1.12) implies}$$

$$v_{2n+1} = A^{2n+1} - \sum_{k=1}^n \binom{2n+1}{k} (rs)^k (r^{2n+1-2k} + s^{2n+1-2k}), \text{ so (1.2) and (1.6) imply}$$

$$v_{2n+1} = A^{2n+1} - \sum_{k=1}^n \binom{2n+1}{k} B^k v_{2n+1-2k}.$$

LEMMA 2.3. $v_p \equiv v_1 \pmod{p}$

PROOF. Lemma 2.2 and (1.11) imply $v_p \equiv A^p \pmod{p}$; (1.13) implies $A^p \equiv A \pmod{p}$; so (1.4) implies $v_p \equiv v_1 \pmod{p}$.

LEMMA 2.4. If $i > j$, then $v_{i+j} = v_i v_j - B^j v_{i-j}$

PROOF. By (1.2) and (1.6),

$$v_i v_j - v_{i+j} = (r^i + s^i)(r^j + s^j) - (r^{i+j} + s^{i+j}) = r^i s^j + r^j s^i - (rs)^j (r^{i-j} + s^{i-j}) = B^j v_{i-j}.$$

LEMMA 2.5. If

$0 < m < n$, $y \equiv z \pmod{p^m}$, $w \equiv x \pmod{p^n}$, and $x \equiv 0 \pmod{p^{n-m}}$,

then $wy \equiv xz \pmod{p^n}$.

PROOF. Hypothesis implies $y = z + ip^m$, $w = x + jp^n$, so $wy \equiv xz + ip^m x \pmod{p^n}$. Hypothesis also implies $p^{n-m} \mid x$, so $wy \equiv xz \pmod{p^n}$.

LEMMA 2.6. If

$k > 0$, $m > 1$, and $v_p^m \equiv v_{p^{m-1}} \pmod{p^m}$, then $v_{kp^m} \equiv v_{kp^{m-1}} \pmod{p^m}$.

PROOF. (Induction on k). Lemma 6 holds trivially for $k=0$, and by hypothesis for $k=1$.

$$v_{(k+1)p^m} = v_{kp^m+p^m} = v_{kp^m} v_{p^m} - B^{p^m} v_{kp^m-p^m} = v_{kp^m} v_{p^m} - B^{p^m} v_{(k-1)p^m} \quad \text{by Lemma 2.4.}$$

Now induction hypothesis and (1.13) imply that

$$v_{(k+1)p^m} \equiv v_{kp^{m-1}} v_{p^{m-1}} - B^{p^{m-1}} v_{(k-1)p^{m-1}} \pmod{p^m}. \quad \text{Now Lemma 2.4 implies}$$

$$v_{(k+1)p^m} \equiv v_{(k+1)p^{m-1}} \pmod{p^m}.$$

LEMMA 2.7. If p is odd and $n > 1$, then $v_{p^n} \equiv v_{p^{n-1}} \pmod{p^n}$.

PROOF. (Induction on n) Lemma 2.7 holds for $n=1$, by Lemma 2.3. Suppose Lemma 2.7 holds for all $m < n$, where $n > 2$. Then Lemma 2.2 implies

$$v_{p^{n-1}} = A^{p^{n-1} \frac{1}{2} (p^{n-1}-1)} \sum_{i=0}^{p^{n-1}-1} (p_i^{n-1}) B^i v_{p^{n-1}-2i}, \text{ also}$$

$$v_{p^n} = A^{p^n \frac{1}{2} (p^n-1)} \sum_{j=0}^{p^n-1} (p_j^n) B^j v_{p^n-2j}.$$

If $p \nmid j$, then (1.15) implies $(p_j^n) \not\equiv 0 \pmod{p^n}$.

Therefore

$$v_{p^n} \equiv A^{p^n \frac{1}{2} (p^{n-1}-1)} \sum_{j=0}^{p^{n-1}-1} (p_{ip}^n) B^{ip} v_{p^n-2ip}.$$

Let $ip = hp^m$, where $p \nmid h$ and $m < n$. Now

$$B^{ip} \equiv B^{hp^m} \equiv B^{hp^{m-1}} \equiv B^i \pmod{p^m}, \text{ by (1.13); also } v_{p^n-2ip} =$$

$$v_{p^{n-2}hp^m} = v_{(p^{n-m}-2h)p^m} \equiv v_{(p^{n-m}-2h)p^{m-1}} \equiv v_{p^{n-1}-2i} \pmod{p^m} \quad \text{by induction hypothesis}$$

and Lemma 2.6. Therefore $B^{ip} v_{p^n-2ip} \equiv B^i v_{p^{n-1}-2i} \pmod{p^m}$. Also

$$(p_{ip}^n) \equiv (p_{hp^m}^n) \equiv (p_{hp^{m-1}}^{n-1}) \equiv (p_i^{n-1}) \equiv (p_h^{n-m}) \pmod{p^n} \quad \text{by Theorem 2.1, and (1.15)}$$

implies

$$(p_h^{n-m}) \equiv 0 \pmod{p^{n-m}}. \quad \text{Therefore Lemma 2.5 implies}$$

$$(p_{ip}^n) B^{ip} v_{p^n-2ip} \equiv (p_i^{n-1}) B^i v_{p^{n-1}-2i} \pmod{p^n}. \quad \text{Now (1.13) implies}$$

$$v_p^n \equiv A p^{n-1} \sum_{i=0}^{1/2(p^{n-1}-1)} \binom{p^{n-1}}{i} B^i v_{p^{n-1}-2i} \pmod{p^n}, \text{ that is,}$$

$$v_p^n \equiv v_{p^{n-1}} \pmod{p^n}.$$

LEMMA 2.8. If $2 \mid A$, then $v_{2^n} \equiv 2 \pmod{2^{n+1}}$ for $n > 0$.

PROOF. (Induction on n) $v_{2^0} = v_1 = A = 2 \pmod{2}$ by (1.4) and hypothesis. Now induction hypothesis implies $v_{2^{n-1}} \equiv 2 \pmod{2^n}$, so (1.14) implies $v_{2^{n-1}}^2 \equiv 4 \pmod{2^{n+1}}$. Hypothesis implies $2 \nmid B$, so $B^{2^{n-1}} \equiv 1 \pmod{2^n}$. Now (1.9) implies $v_{2^n} = v_{2^{n-1}}^2 - 2B^{2^{n-1}} \equiv 4 - 2(1) \equiv 2 \pmod{2^{n+1}}$.

LEMMA 2.9. If $2 \nmid AB$, then $v_{2^n} \equiv -1 \pmod{2^{n+1}}$ for $n > 0$.

PROOF. (Induction on n) $v_{2^0} = v_1 = A \equiv -1 \pmod{2}$ by (1.4) and hypothesis. Now induction hypothesis implies $v_{2^{n-1}} \equiv -1 \pmod{2^n}$, so (1.14) implies $v_{2^{n-1}}^2 \equiv 1 \pmod{2^{n+1}}$. Again, B odd implies $B^{2^{n-1}} \equiv 1 \pmod{2^n}$, so (1.9) implies $v_{2^n} = v_{2^{n-1}}^2 - 2B^{2^{n-1}} \equiv 1 - 2(1) \equiv -1 \pmod{2^{n+1}}$.

LEMMA 2.10. If $2 \mid B$, then $v_{2^n} \equiv 1 \pmod{2^{n+1}}$ for $n > 0$.

PROOF. (Induction on n) Hypothesis implies A is odd, so (1.4) implies

$$v_{2^0} = v_1 = A \equiv 1 \pmod{2}. \text{ By hypothesis, } B \equiv 0 \pmod{2}, \text{ so } B^{2^{n-1}} \equiv 0 \pmod{2^{2^{n-1}}}. \text{ Since } 2^{n-1} > n \text{ for } n > 1, \text{ we have } B^{2^{n-1}} \equiv 0 \pmod{2^n}. \text{ By}$$

induction hypothesis, $v_{2^{n-1}} \equiv 1 \pmod{2^n}$, so (1.14) implies $v_{2^{n-1}}^2 \equiv 1 \pmod{2^{n+1}}$.

$$\text{Now (1.9) implies } v_{2^n} = v_{2^{n-1}}^2 - 2B^{2^{n-1}} \equiv 1 - 2(0) \equiv 1 \pmod{2^{n+1}}.$$

LEMMA 2.11. $v_{2^n} \equiv v_{2^{n-1}} \pmod{2^n}$

PROOF. Lemmas 2.8, 2.9, 2.10 imply $v_{2^{n-1}} \equiv t \pmod{2^n}$, $v_{2^n} \equiv t \pmod{2^{n+1}}$,

where $t = 2$ or ± 1 . Therefore $v_{2^n} \equiv t \equiv v_{2^{n-1}} \pmod{2^n}$.

THEOREM 2.2. In $n \geq 1$, then $v_p^n \equiv v_p^{n-1} \pmod{p^n}$.

PROOF. Follows from Lemmas 2.7 and 2.11.

THEOREM 2.3. If $n \geq 1$ and $k \geq 0$, then $v_{kp}^n \equiv v_{kp}^{n-1} \pmod{p^n}$.

PROOF. Follows from Theorem 2.2 and Lemma 2.6.

LEMMA 2.12. $u_{2n+1} = D^n - \sum_{k=1}^n \binom{2n+1}{k} (-B)^k u_{2n+1-2k}$

PROOF. (1.1) and (1.7) imply $D^{1/2} u_{2n+1} = r^{2n+1} - s^{2n+1} =$

$(r-s)^{2n+1} - \sum_{k=1}^{2n} (-1)^k \binom{2n+1}{k} r^{2n+1-k} s^k$, so (1.7) implies

$$D^{1/2} u_{2n+1} = D^{n+1/2} - \left\{ \sum_{k=1}^n (-1)^k \binom{2n+1}{k} r^{2n+1-k} s^k + \sum_{j=n+1}^{2n} (-1)^j \binom{2n+1}{j} r^{2n+1-j} s^j \right\}$$

Setting $j = 2n+1-k$ in the second sum, we obtain

$$D^{1/2} u_{2n+1} = D^{n+1/2} - \left\{ \sum_{k=1}^n (-1)^k \binom{2n+1}{k} r^{2n+1-k} s^k - \sum_{k=1}^n (-1)^{2n-k} \binom{2n+1}{2n+1-k} r^k s^{2n+1-k} \right\}$$

$$= D^{n+1/2} - \sum_{k=1}^n (-1)^k (rs)^k \binom{2n+1}{k} (r^{2n+1-2k} - s^{2n+1-2k}) \quad \text{by (1.12)}$$

$$= D^{n+1/2} - D^{1/2} \sum_{k=1}^n \binom{2n+1}{k} (-B)^k u_{2n+1-2k} \quad \text{by (1.6) and (1.7), so}$$

$$u_{2n+1} = D^n - \sum_{k=1}^n \binom{2n+1}{k} (-B)^k u_{2n+1-2k}.$$

LEMMA 2.13. If p is odd, then $u_p \equiv \left(\frac{D}{p}\right) \pmod{p}$.

PROOF. Follows from Lemma 2.12, (1.11), and Euler's criterion.

LEMMA 2.14. If p is odd, $p \nmid D$, and $n \geq 1$, then $D^{1/2} \mathcal{J}(p^n) \equiv \left(\frac{D}{p}\right) \pmod{p^n}$.

PROOF. (Induction on n) Lemma 2.14 holds for $n=1$ by Euler's criterion. Let $\left(\frac{D}{p}\right) = t = \pm 1$. Now induction hypothesis implies

$$D^{1/2} \mathcal{J}(p^n) \equiv t \pmod{p^n}, \text{ so } D^{1/2} \mathcal{J}(p^n) = t + ip^n. \quad D^{1/2} \mathcal{J}(p^{n+1}) = (D^{1/2} \mathcal{J}(p^n))^p$$

$$= (t + ip^n)^p = t^p + p t^{p-1} (ip^n) + \sum_{j=2}^p \binom{p}{j} t^{p-j} (ip^n)^j. \quad \text{Now } t^p = t,$$

and $p^{n+1} \nmid p^{nj}$ for $j > 2$, so $\frac{1}{2} \delta(p^{n+1}) \equiv t \pmod{p^{n+1}}$.

LEMMA 2.15. If $k > 0$, $m > 1$, p is odd, $t = (\frac{D}{p})$, and

$$u_p^m \equiv t u_{p^{m-1}} \pmod{p^m}, \text{ then } u_{kp^m} \equiv t u_{kp^{m-1}} \pmod{p^m}.$$

PROOF. (Induction on k) Lemma 2.15 is trivially true for $k=0$, and is true by hypothesis for $k=1$. Now (1.10) implies

$$2u_{(k+1)p^m} = 2u_{kp^m+p^m} = u_{kp^m}^v u_p^v + u_p^v u_{kp^m}^v. \text{ By induction hypothesis,}$$

$$u_{kp^m} \equiv t u_{kp^{m-1}} \pmod{p^m}, \text{ and } u_p^v \equiv t u_{p^{m-1}} \pmod{p^m};$$

Theorem 2.2 implies $v_{kp^m} \equiv v_{kp^{m-1}} \pmod{p^m}$ for $k > 0$. Therefore

$$2u_{(k+1)p^m} \equiv t u_{kp^{m-1}}^v u_{p^{m-1}}^v + t u_{p^{m-1}}^v u_{kp^{m-1}}^v \equiv 2t u_{kp^m} \pmod{p^m}. \text{ Since}$$

p is odd, we have $u_{(k+1)p^m} \equiv t u_{kp^m} \pmod{p^m}$.

LEMMA 2.16. If p is odd, $p \nmid D$, and $n > 1$, then $u_p^n \equiv (\frac{D}{p}) u_{p^{n-1}} \pmod{p^n}$.

PROOF. (Induction on n) Lemma 2.16 is true for $n = 1$ by Lemma 2.13.

$$\text{Lemma 2.12 implies } u_{p^{n-1}} = \frac{1}{2} (p^{n-1}-1) - \frac{1}{2} \sum_{i=1}^{p^{n-1}-1} \binom{p^{n-1}-1}{p_i^{n-1}} (-B)^i u_{p^{n-1}-2i}$$

$$\text{also } u_p^n = \frac{1}{2} (p^n-1) - \frac{1}{2} \sum_{j=1}^{p^n-1} \binom{p^n-1}{p_j^n} (-B)^j u_{p^n-2j}. \text{ If } p \nmid j, \text{ then (1.15) implies}$$

$$\binom{p^n}{p_j^n} \equiv 0 \pmod{p^n}. \text{ Therefore}$$

$$u_p^n \equiv \frac{1}{2} (p^n-1) - \frac{1}{2} \sum_{i=1}^{p^{n-1}-1} \binom{p^n}{ip^n} (-B)^i u_{p^n-2ip} \pmod{p^n}.$$

Let $ip = hp^m$ where $p \nmid h$ and $m < n$. Let $t = (\frac{D}{p})$.

Now $(-B)^{ip} \equiv (-B)^{hp^m} \equiv (-B)^{hp^{m-1}} \equiv (-B)^i \pmod{p^m}$ by (1.13). By induction

hypothesis and Lemma 2.15, we have

$$u_{p^n-2ip} \equiv u_{p^n-2hp^m} \equiv u_{(p^{n-m}-2h)p^m} \equiv t u_{(p^{n-m}-2h)p^{m-1}} \equiv t u_{p^{n-1}-2hp^{m-1}} \equiv$$

$$tu_{p^{n-1-2i}} \pmod{p^m}. \quad \text{Therefore } (-B)^i u_{p^{n-2i}} \equiv t(-B)^i u_{p^{n-1-2i}} \pmod{p^m}.$$

As in the proof of Lemma 2.7, we have

$$\binom{p}{ip} (-B)^i u_{p^{n-2i}} \equiv t \binom{p^{n-1}}{i} (-B)^i u_{p^{n-1-2i}} \pmod{p^n}. \quad \text{Therefore}$$

$$u_{p^n} \equiv D^{1/2} (p^{n-1})^{-1/2} - t(D^{1/2} (p^{n-1-1})^{-1/2} - u_{p^{n-1}}) \pmod{p^n}, \text{ that is}$$

$$u_{p^n} \equiv tu_{p^{n-1}} + D^{1/2} (p^{n-1-1})^{-1/2} (D^{1/2} (p^{n-p^{n-1}})^{-1/2} - t) \pmod{p^n}. \quad \text{Since}$$

$$\not\equiv (p^n) = p^{n-p^{n-1}}, \text{ Lemma 2.14 implies } u_{p^n} \equiv \left(\frac{D}{p}\right) u_{p^{n-1}} \pmod{p^n}.$$

LEMMA 2.17. If $n > 1$ and D is odd, then $u_{2^n} \equiv (-1)^B u_{2^{n-1}} \pmod{2^n}$.

PROOF. By hypothesis and by the definitions of A , B , and D , A must be odd. If B

is odd, then Lemma 2.9 implies $v_{2^{n-1}} \equiv -1 \pmod{2^n}$; if B is even then Lemma 2.10 implies $v_{2^{n-1}} \equiv 1 \pmod{2^n}$. Therefore, in either case, $v_{2^{n-1}} \equiv (-1)^B \pmod{2^n}$.

Now (1.8) implies $u_{2^n} \equiv (-1)^B u_{2^{n-1}} \pmod{2^n}$.

THEOREM 2.4. If $n > 1$ and $p \nmid D$, then $u_{p^n} \equiv tu_{p^{n-1}} \pmod{p^n}$, where

$$t = \begin{cases} \left(\frac{D}{p}\right) & \text{if } p \text{ is odd} \\ (-1)^B & \text{if } p = 2 \end{cases}$$

PROOF. Follows from Lemmas 2.16 and 2.17.

THEOREM 2.5. If $k > 0$, $n > 1$, and $p \nmid D$, then $u_{kp^n} \equiv tu_{kp^{n-1}} \pmod{p^n}$.

where t is defined as in Theorem 2.4.

PROOF. Follows from Lemma 2.15 and Theorem 2.4.

Concluding Remarks. Let t be defined as in Theorem 2.4 as a result of Theorems 2.3 and 2.5, the sequences v_{kp^n} , $t^n u_{kp^n}$ determine p -adic integers for each $k > 0$.

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