COEFFICIENT ESTIMATES FOR SOME CLASSES OF p -VALENT FUNCTIONS

M.K. AOUF

Department of Mathematics
Faculty of Science
Mansoura University
Mansoura, EGYPT

(Received January 6, 1985 and in revised form May 7, 1985)

ABSTRACT. Let A_p , where p is a positive integer, denote the class of functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \text{ which are analytic in } U = \{z: |z| < 1\}.$$

For $0 < \lambda \leq 1$, $|\alpha| < \frac{\pi}{2}$, $0 \leq \beta < p$, let $F_{\lambda}(\alpha, \beta, p)$ denote the class of functions $f(z) \in A_p$ which satisfy the condition

$$\left| \frac{H(f(z)) - 1}{H(f(z)) + 1} \right| < \lambda \text{ for } z \in U,$$

$$\text{where } H(f(z)) = \frac{\frac{iazf'(z)}{e^{f(z)}} - \beta \cos \alpha - ip \sin \alpha}{(p - \beta) \cos \alpha}.$$

Also let $C_{\lambda}(b, p)$, where p is a positive integer, $0 < \lambda < 1$, and $b \neq 0$ is any complex number, denote the class of functions $g(z) \in A_p$ which satisfy the condition

$$\left| \frac{H(g(z)) - 1}{H(g(z)) + 1} \right| < \lambda \text{ for } z \in U, \text{ where}$$

$$H(g(z)) = 1 + \frac{1}{pb} \left(1 + \frac{zg''(z)}{g'(z)} - p \right).$$

In this paper we obtain sharp coefficient estimates for the above mentioned classes.

KEY WORDS AND PHRASES. p -valent, starlike, convex, spirallike functions.

1980 AMS SUBJECT CLASSIFICATION CODES. 30A32, 30A36.

1. INTRODUCTION.

Let A_p , where p is a positive integer, denote the class of functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \text{ which are analytic in } U = \{z: |z| < 1\}. \text{ We use } \Omega_{\lambda}, 0 < \lambda \leq 1,$$

to denote the class of analytic functions $w(z)$ in U satisfying the conditions $w(0) = 0$ and $|w(z)| < \lambda$, $0 < \lambda \leq 1$.

Padmanabhan introduced the class of starlike functions of bounded order λ , $0 < \lambda \leq 1$, defined as follows [11]:

DEFINITION 1. A function $f \in A_1$ and satisfying

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1} \right| < \lambda \quad (1.1)$$

for a given λ , $0 < \lambda \leq 1$, $|z| < 1$ is said to be starlike of bounded order λ in $|z| < 1$ and this class is denoted $S(\lambda)$, the class of all such functions for a given λ .

Let $F(\alpha, \beta, p)$ ($|\alpha| < \frac{\pi}{2}$, $0 \leq \beta < p$) denote the class of functions $f(z) \in A_p$ and for which there exists a $\rho = \rho(f)$ such that

$$\operatorname{Re} \left\{ e^{i\alpha} \frac{zf'(z)}{f(z)} \right\} > \beta \cos \alpha \quad (1.2)$$

and

$$\int_0^{2\pi} \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} d\theta = 2\pi p \quad \text{for } z = re^{i\theta}, \rho < r < 1. \quad (1.3)$$

Functions in $F(\alpha, \beta, p)$ are called p -valent α -spirallike functions of order β . The class $F(\alpha, \beta, p)$ was introduced by Patil and Thakare [12].

In this paper we use a method of Clunie [3] to obtain sharp bounds for the coefficients of functions $F_\lambda(\alpha, \beta, p)$ and $C_\lambda(b, p)$, where p is a positive integer, $0 < \lambda \leq 1$, $|\alpha| < \frac{\pi}{2}$, $0 \leq \beta < p$, and b is any complex number, where $F_\lambda(\alpha, \beta, p)$ and $C_\lambda(b, p)$ are defined as follows:

DEFINITION 2. For $0 < \lambda \leq 1$, $|\alpha| < \frac{\pi}{2}$, and $0 \leq \beta < p$, let $F_\lambda(\alpha, \beta, p)$ denote the class of functions $f(z) \in A_p$ which satisfy the condition

$$\left| \frac{H(f(z)) - 1}{H(f(z)) + 1} \right| < \lambda \quad (1.4)$$

for $z \in U$, where

$$H(f(z)) = \frac{e^{i\alpha} \frac{zf'(z)}{f(z)} - \beta \cos \alpha - ip \sin \alpha}{(p - \beta) \cos \alpha}. \quad (1.5)$$

DEFINITION 3. For p is a positive integer, $0 < \lambda \leq 1$, and $b \neq 0$ is any complex number, let $C_\lambda(p, b)$ denote the class of functions $g(z) \in A_p$ which satisfy the condition

$$\left| \frac{H(g(z)) - 1}{H(g(z)) + 1} \right| < \lambda \quad (1.6)$$

for $z \in U$,

$$\text{where} \quad H(g(z)) = 1 + \frac{1}{pb} \left(1 + \frac{zg''(z)}{g'(z)} - p \right). \quad (1.7)$$

We note that by giving specific values to λ , α , β , p and b , we obtain the following important subclasses studied by various authors in earlier papers:

(1) $F_1(0, 0, 1) = S^*$ and $C_1(1, 1) = C$, are respectively the well-known classes of starlike functions and convex functions, $F_1(0, \beta, 1) = S_\beta$ and $C_1(1 - \beta, 1) = C_\beta$, $0 \leq \beta < 1$, are respectively the classes of starlike functions of order β and convex functions of order β introduced by Robertson [14], $F_\lambda(0, 0, 1) = S(\lambda)$ and $C_\lambda(1, 1) = C(\lambda)$, is the class of functions g for which $zg'(z) \in S(\lambda)$.

(2) $F_1(\alpha, 0, 1) = S^\alpha$ and $C_1(\cos \alpha e^{-i\alpha}, 1) = C^\alpha$, $|\alpha| < \frac{\pi}{2}$, are respectively the class of α -spirallike functions introduced by Špaček [18] and the class of functions g for which $zg'(z)$ is α -spirallike introduced by Robertson [15], $F_1(\alpha, \beta, 1) = S_\beta^\alpha$ and $C_1[(1-\beta) \cos \alpha e^{-i\alpha}, 1] = C_\beta^\alpha$, $|\alpha| < \frac{\pi}{2}$, $0 < \beta \leq 1$, are respectively the class of α -spirallike functions of order β introduced by Libera [8] and the class of functions g for which $zg'(z)$ is α -spirallike of order β by Chichra [2] and Sizuk [17].

(3) $C_1(b, 1) = C(b)$ is the class of functions $g \in A_1$ satisfying

$$\operatorname{Re}\left\{1 + \frac{1}{b} \frac{zg''(z)}{g'(z)}\right\} > 0$$

introduced by Wiatrowski [19] and studied by [9] and [10].

(4) $F_1(0, 0, p) = S(p)$, $C_1(1, p) = C(p)$, $F_1(0, \beta, p) = S_\beta(p)$ and $C_1[(1-\frac{\beta}{p}), p] = C_\beta(p)$, $0 \leq \beta < p$, are respectively the classes of p -valent starlike functions, p -valent convex functions, p -valent starlike functions of order β and p -valent convex functions of order β considered by Goodman [6] and the class $S_\beta(p)$ investigated by Goluzina [5].

(5) $F_1(\alpha, 0, p) = S^\alpha(p)$ and $C_1(\cos \alpha e^{-i\alpha}, p)$, $|\alpha| < \frac{\pi}{2}$, are respectively the class of p -valent α -spirallike functions and the class of p -valent functions $g \in A_p$ satisfying

$$\operatorname{Re} e^{i\alpha} \left(1 + \frac{zg''(z)}{g'(z)}\right) > 0, \quad z \in U$$

i.e., the class of p -valent functions g for which $\frac{zg'(z)}{p}$ is p -valent α -spirallike.

(6) $F_1(\alpha, \beta, p) = F(\alpha, \beta, p)$ and $C_1[(1-\frac{\beta}{p}) \cos \alpha e^{-i\alpha}, p]$, $|\alpha| < \frac{\pi}{2}$, $0 \leq \beta < p$, is the class of p -valent functions g for which $\frac{zg'(z)}{p}$ is p -valent α -spirallike of order β .

(7) $C_1(b, p)$, is the class of functions $g \in A_p$ satisfying

$$\operatorname{Re} \left\{p + \frac{1}{b} \left(1 + \frac{zg''(z)}{g'(z)} - p\right)\right\} > 0, \quad z \in U,$$

the class $C(b, p)$ was introduced by the author [1].

(8) $F_\lambda(\alpha, \beta, 1) = F_\lambda(\alpha, \beta)$, is the class of functions investigated by Gopalakrishna and Umarani [7].

(9) $C_1[(1-\frac{\beta}{p}) \cos \alpha e^{i\alpha}, p]$, $|\alpha| < \frac{\pi}{2}$, $0 \leq \beta < p$, is the class of p -valent functions $g(z)$ for which $\frac{zg'(z)}{p} \in F_\lambda(\alpha, \beta, p)$.

We state the following lemma that is needed in our investigation.

LEMMA 1[11]. Let $f(z)$ be analytic for $|z| < 1$ and let $f(0) = 0$. Then $f(z) \in S(\lambda)$ if and only if

$$f(z) = z \exp \left[-2 \int_0^z \frac{\phi(t)}{1 + t\phi(t)} dt\right],$$

where $\phi(z)$ is analytic and satisfies $|\phi(z)| \leq \lambda$, $0 < \lambda \leq 1$, for $|z| < 1$.

In the rest of the paper we always assume that p is a positive integer,

$0 < \lambda \leq 1$, $|\alpha| < \frac{\pi}{2}$, $0 \leq \beta < p$, and $b \neq 0$ is any complex number.

2. REPRESENTATION FORMULAS FOR THE CLASS $F_\lambda(\alpha, \beta, p)$.

LEMMA 2. $f(z) \in F_\lambda(\alpha, \beta, p)$ if and only if for $z \in U$

$$e^{i\alpha} \frac{zf'(z)}{f(z)} = \cos \alpha \left\{ \frac{p-(p-2\beta)w(z)}{1+w(z)} \right\} + ip \sin \alpha, \quad (2.1)$$

$w \in \Omega_\lambda$.

PROOF. If $f(z)$ is given by (2.1), then

$$\begin{aligned} H(f(z)) &= \frac{e^{i\alpha} \frac{zf'(z)}{f(z)} - \beta \cos \alpha - ip \sin \alpha}{(p-\beta) \cos \alpha} \\ &= \frac{1-w(z)}{1+w(z)} \end{aligned}$$

so that $\frac{H(f(z)) - 1}{H(f(z)) + 1} = -w(z)$

and so (1.4) holds. Thus $f(z) \in F_\lambda(\alpha, \beta, p)$.

Conversely, if $f(z) \in F_\lambda(\alpha, \beta, p)$, then (1.4) holds.

Defining $w(z) = \frac{1-H(f(z))}{1+H(f(z))}$ we obtain (2.1) and the proof is complete.

LEMMA 3. $f(z) \in F_\lambda(\alpha, \beta, p)$ if and only if

$$f(z) = z^p \left[\frac{f_1(z)}{z} \right]^p \quad (2.2)$$

for some $f_1 \in F_\lambda(\alpha, \frac{\beta}{p}, 1)$.

PROOF. Let $f(z) = z^p \left[\frac{f_1(z)}{z} \right]^p$ for $f_1(z) = z + \sum_{n=2}^{\infty} c_n z^n \in F_\lambda(\alpha, \frac{\beta}{p}, 1)$, $z \in U$.

By direct computation, we obtain

$$\frac{e^{i\alpha} \frac{zf_1'(z)}{f_1(z)} - \beta \cos \alpha - ip \sin \alpha}{(p-\beta) \cos \alpha} = \frac{e^{i\alpha} \frac{zf'(z)}{f(z)} - \beta \cos \alpha - ip \sin \alpha}{(1 - \frac{\beta}{p}) \cos \alpha}$$

and the result follows from (1.4).

In a similar way we can prove the following lemma:

LEMMA 4. $f(z) \in F_\lambda(\alpha, \beta, p)$ if and only if

$$f(z) = z^p \left[\frac{f_2(z)}{z} \right]^{(p-\beta) \cos \alpha} e^{-i\alpha} \quad (2.3)$$

for some $f_2 \in S(\lambda)$.

An immediate consequence of lemmas 1 and 4 is

THEOREM 1. $f(z) \in F_\lambda(\alpha, \beta, p)$ if and only if

$$f(z) = z^p \exp[-2(p-\beta) \cos \alpha e^{-i\alpha} \int_0^z \frac{\phi(t)}{1+t\phi(t)} dt] \quad (2.4)$$

where $\phi(z)$ is analytic and satisfies $|\phi(z)| \leq \lambda$, $0 < \lambda \leq 1$, for $|z| < 1$.

3. COEFFICIENT ESTIMATES FOR THE CLASS $F_\lambda(\alpha, \beta, p)$.

LEMMA 5. If integers p and m are greater than zero, $0 \leq \beta < p$ and $|\alpha| < \frac{\pi}{2}$, then

$$\begin{aligned}
& \prod_{j=0}^{m-1} \frac{\lambda^2 |2(p-\beta) \cos \alpha e^{-i\alpha} + j|^2}{(j+1)^2} = \frac{\cos^2 \alpha}{m^2} \{4 \lambda^2 (p-\beta)^2 \\
& + \prod_{k=1}^m [\lambda^2 (2p-2\beta+k)^2 + \lambda^2 k^2 \tan^2 \alpha - k^2 \sec^2 \alpha] \times \\
& \prod_{j=0}^{k-1} \frac{\lambda^2 |2(p-\beta) \cos \alpha e^{-i\alpha} + j|^2}{(j+1)^2} \} . \quad (3.1)
\end{aligned}$$

PROOF. We prove the lemma by induction on m . For $m = 1$, (3.1) is easily verified directly.

Next suppose that (3.1) is true for $m = q-1$. We have

$$\begin{aligned}
& \frac{\cos^2 \alpha}{q^2} \{4 \lambda^2 (p-\beta)^2 + \prod_{k=1}^{q-1} [\lambda^2 (2p-2\beta+k)^2 + \lambda^2 k^2 \tan^2 \alpha \\
& - k^2 \sec^2 \alpha] \cdot \prod_{j=0}^{k-1} \frac{\lambda^2 |2(p-\beta) \cos \alpha e^{-i\alpha} + j|^2}{(j+1)^2} \} \\
& = \frac{\cos^2 \alpha}{q^2} \{4 \lambda^2 (p-\beta)^2 + \prod_{k=1}^{q-2} [\lambda^2 (2p-2\beta+k)^2 \\
& + \lambda^2 k^2 \tan^2 \alpha - k^2 \sec^2 \alpha] \prod_{j=0}^{k-1} \frac{\lambda^2 |2(p-\beta) \cos \alpha e^{-i\alpha} + j|^2}{(j+1)^2} \\
& + [\lambda^2 (2p-2\beta+q-1)^2 + \lambda^2 (q-1)^2 \tan^2 \alpha - \\
& (q-1)^2 \sec^2 \alpha] \prod_{j=0}^{q-2} \frac{\lambda^2 |2(p-\beta) \cos \alpha e^{-i\alpha} + j|^2}{(j+1)^2} \} \\
& = \prod_{j=0}^{q-2} \frac{\lambda^2 |2(p-\beta) \cos \alpha e^{-i\alpha} + j|^2}{(j+1)^2} \times \\
& \{ \frac{\lambda^2 (2p-2\beta+q-1)^2 \cos^2 \alpha + \lambda^2 (q-1)^2 \sin^2 \alpha}{q^2} \} \\
& = \prod_{j=0}^{q-1} \frac{\lambda^2 |2(p-\beta) \cos \alpha e^{-i\alpha} + j|^2}{(j+1)^2}
\end{aligned}$$

Thus (3.1) holds for $m=q$ which proves lemma 5.

THEOREM 2. If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in F_{\lambda}(\alpha, \beta, p)$, then

$$|a_n| \leq \prod_{k=0}^{n-(p+1)} \frac{\lambda |2(p-\beta) \cos \alpha e^{-i\alpha} + k|}{k+1} \quad (3.2)$$

for $n \geq p+1$ and these bounds are sharp for all admissible α, β and λ for each n .

PROOF. As $f \in F_{\lambda}(\alpha, \beta, p)$, from Lemma 2, we have

$$\begin{aligned}
& \{e^{i\alpha} \sec \alpha z f'(z) + (p-2\beta-ip \tan \alpha) f(z)\} w(z) \\
& = (p+ip \tan \alpha) f(z) - e^{-i\alpha} \sec \alpha z f'(z)
\end{aligned}$$

for $z \in U$, $w \in \Omega_{\lambda}$. Hence we have

$$\begin{aligned}
& \sum_{k=0}^{\infty} [(p+k) e^{i\alpha} \sec \alpha + (p-2\beta-ip \tan \alpha)] a_{p+k} z^k w(z) \\
& = \sum_{k=0}^{\infty} [p + ip \tan \alpha - (p+k) e^{i\alpha} \sec \alpha] a_{p+k} z^k \quad (3.3)
\end{aligned}$$

where $a_p = 1$ and $w(z) = \sum_{k=0}^{\infty} b_{k+1} z^{k+1}$.

Equating coefficients of z^m on both sides of (3.3), we obtain

$$\sum_{k=0}^{m-1} \{(p+k)e^{i\alpha} \sec \alpha + (p-2\beta-ip \tan \alpha)\} a_{p+k} b_{m-k} \\ = \{p+ip \tan \alpha - (p+m)e^{i\alpha} \sec \alpha\} a_{p+m};$$

which shows that a_{p+m} on right hand side depends only on

$$a_p, a_{p+1}, \dots, a_{p+(m-1)}$$

of left-hand side. Hence we can write

$$\sum_{k=0}^{m-1} \{[(p+k)e^{i\alpha} \sec \alpha + (p-2\beta-ip \tan \alpha)] a_{p+k} z^k\} w(z) \\ = \sum_{k=0}^m [p + ip \tan \alpha - (p+k)e^{i\alpha} \sec \alpha] a_{p+k} z^k + \sum_{k=m+1}^{\infty} A_k z^k \quad (3.4)$$

for $m = 1, 2, 3, \dots$ and a proper choice of A_k ($k \geq 0$).

Denoting the right member of (3.4) by $G(z)$ and the factor multiplying $w(z)$ in the left member of (3.4) by $F(z)$, (3.4) assumes the form

$$G(z) = F(z) w(z) \quad \text{for } z \in U.$$

Since $|w(z)| < \lambda$ for $z \in U$ this yields for $0 < r < 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} |G(re^{i\theta})|^2 d\theta \leq \frac{\lambda^2}{2\pi} \cdot \int_0^{2\pi} |F(re^{i\theta})|^2 d\theta,$$

hence, using the definitions of $G(z)$ and $F(z)$

$$\sum_{k=0}^m |p+ip \tan \alpha - (p+k)e^{i\alpha} \sec \alpha|^2 |a_{p+k}|^2 r^{2k} \\ + \sum_{k=m+1}^{\infty} |A_k|^2 r^{2k} \leq \\ \lambda^2 \left\{ \sum_{k=0}^{m-1} |(p+k)e^{i\alpha} \sec \alpha + (p-2\beta-ip \tan \alpha)|^2 |a_{p+k}|^2 r^{2k} \right\}. \quad (3.5)$$

Setting $r \rightarrow 1$ in (3.5), the inequality (3.5) may be written as

$$\sum_{k=0}^{m-1} \{\lambda^2 |(p+k)e^{i\alpha} \sec \alpha + (p-2\beta-ip \tan \alpha)|^2 - \\ |p+ip \tan \alpha - (p+k)e^{i\alpha} \sec \alpha|^2\} |a_{p+k}|^2 \\ \geq |p+ip \tan \alpha - (p+m)e^{i\alpha} \sec \alpha|^2 |a_{p+m}|^2. \quad (3.6)$$

Simplification of (3.6) leads to

$$|a_{p+m}|^2 \leq \frac{\cos^2 \alpha}{m^2} \cdot \sum_{k=0}^{m-1} \{\lambda^2 (2p-2\beta+k)^2 + \\ \lambda^2 k^2 \tan^2 \alpha - k^2 \sec^2 \alpha\} |a_{p+k}|^2. \quad (3.7)$$

Replacing $p+m$ by n in (3.7), we are led to

$$|a_n|^2 \leq \frac{\cos^2 \alpha}{(n-p)^2} \cdot \sum_{k=0}^{n-(p+1)} \{\lambda^2 (2p-2\beta+k)^2 + \\ \lambda^2 k^2 \tan^2 \alpha - k^2 \sec^2 \alpha\} |a_{p+k}|^2 \quad (3.8)$$

where $n \geq p+1$.

For $n = p + 1$, (3.8) reduces to

$$|a_{p+1}|^2 \leq 4(p-\beta)^2 \lambda^2 \cos^2 \alpha$$

or

$$|a_{p+1}| \leq 2(p-\beta) \lambda \cos \alpha \quad (3.9)$$

which is equivalent to (3.2).

To establish (3.2) for $n > p+1$, we will apply induction argument.

Fix n , $n \geq p + 2$, and suppose (3.2) holds for $k = 1, 2, \dots, n-(p+1)$. Then

$$|a_n|^2 \leq \frac{\cos^2 \alpha}{(n-p)^2} \{ 4\lambda^2 (p-\beta)^2 + \sum_{k=0}^{n-(p+1)} [\lambda^2 (2p-2\beta+k)^2 + \lambda^2 k^2 \tan^2 \alpha - k^2 \sec^2 \alpha] \times \sum_{j=0}^{k-1} \frac{\lambda^2 |2(p-\beta) \cos \alpha e^{-i\alpha} + j|^2}{(j+1)^2} \} \quad (3.10)$$

Thus from (3.8), (3.10) and Lemma 5 with $m = n - p$, we obtain

$$|a_n|^2 \leq \sum_{j=0}^{n-(p+1)} \frac{\lambda^2 |2(p-\beta) \cos \alpha e^{-i\alpha} + j|^2}{(j+1)^2}.$$

This completes the proof of Theorem 2.

Equality holds in (3.2) for $n \geq p + 1$ for the function $f(z) \in A_p$ defined by (2.1) with $w(z) = \lambda z$.

REMARK ON THEOREM 2. For various choices of the parameters, known results can be regained: [7], [8], [12], [13], [14], [16], [20].

In a similar way we can prove the following: Lemma 6, 7, and Theorem 3 for functions in $C_\lambda(b, p)$.

4. REPRESENTATION FORMULAS FOR THE CLASS $C_\lambda(b, p)$

LEMMA 6. $g(z) \in C_\lambda(b, p)$ if and only if for $z \in U$

$$(i) \quad \frac{zg''(z)}{g'(z)} = \frac{(p-1) + (p-2pb-1)w(z)}{1+w(z)}, \quad w \in \Omega_\lambda. \quad (4.1)$$

$$(ii) \quad g'(z) = pz^{p-1} \left[\frac{g_1(z)}{z} \right]^{pb} \quad (4.2)$$

for some $g_1 \in S(\lambda)$.

$$(iii) \quad g'(z) = pz^{p-1} \exp \left[-2pb \int_0^z \frac{\phi(t)}{1+t} dt \right], \quad (4.3)$$

where $\phi(z)$ is analytic and satisfies $|\phi(z)| \leq \lambda$, $0 < \lambda < 1$, for $|z| < 1$.

5. COEFFICIENT ESTIMATES FOR THE CLASS $C_\lambda(b, p)$.

LEMMA 7. If integers p and m are greater than zero; $b \neq 0$ and complex, then

$$\sum_{j=0}^{m-1} \frac{\lambda^2 |2pb+j|^2}{(j+1)^2} = \frac{1}{m^2} \{ 4 p^2 |b|^2 \cdot \lambda^2 + \sum_{k=1}^{m-1} (k^2 (\lambda^2 - 1) + 4p^2 |b|^2 \lambda^2 + 4pk \operatorname{Re}\{b\} \lambda^2) \sum_{j=0}^{k-1} \frac{\lambda^2 |2pb+j|^2}{(j+1)^2} \}. \quad (5.1)$$

THEOREM 3. If $g(z) = z^p + \sum_{n=p+1}^{\infty} d_n z^n \in C_{\lambda}(b, p)$, then

$$|d_n| \leq \frac{p}{n} \cdot \prod_{k=0}^{n-(p+1)} \frac{\lambda |2pb+k|}{(k+1)} \quad (5.2)$$

for $n \geq p+1$. Equality holds in (5.2) for the function $g(z) \in A_p$ defined by (4.1) with $w(z) = \lambda z$.

ACKNOWLEDGEMENT. In conclusion, I would like to thank Professor Dr. D. K. Thomas for his kind encouragement and helpful guidance in preparing this paper. Also the author is thankful to professor Dr. S. M. Shah for reading the manuscript and for helpful suggestions.

REFERENCES

1. AOUF, M.K., p -Valent Classes Related to Convex Functions of Complex Order, to appear, Rocky Mountain J. of Maths.
2. CHICHRA, P.N., Regular Functions $f(z)$ for which $zf'(z)$ is α -spirallike, Proc. Amer. Math. Soc. 49(1975), 151-160.
3. CLUNIE, J. On Meromorphic Schlicht Functions, J. London Math. Soc. 34(1959), 215-216.
4. CLUNIE, J. and KEOGH, F. R., On Starlike and Convex Schlicht Functions, J. London Math. Soc. 35 (1960), 229-236.
5. GOLUZINA, E.G., On the Coefficients of a Class of Functions, Regular in a Disk and having an Integral Representation in it, J. of Soviet Math. (2) 6(1974), 606-617.
6. GOODMAN, A. W., On the Schwarz-Chistoffel Transformation and p -Valent Functions, Trans. Amer. Math. Soc. 68(1950), 204-223.
7. GOPALAKRISHNA, H.S. and UMARANI, P.G., Coefficients Estimates for Some Classes of Spiral-like Functions, Indian J. Pure and Appl. Math. 11(8)(1980), 1011-1017.
8. LIBERA, R. J., Univalent α -spiral Functions, Canad. J. Math. 19(1967), 449-456.
9. NASR, M. A. and AOUF, M.K. On Convex Functions of Complex Order, Mansoura Science Bull. Egypt 9(1982), 565-582.
10. NASR, M.A. and AOUF, M. K., Radius of Convexity for the Class of Starlike Functions of Complex Order, to appear, Assiut Univ. Bull. of the Faculty of Science Section (A).
11. PADMANABHAN, K. S., On Certain Classes of Starlike Functions in the Unit Disk, J. Indian Math. Soc. 32(1968), 89-103.
12. PATIL, D.A. and THAKARE, N. K., On Coefficient Bound of p -Valent λ -spiral Functions of Order α , Indian J. Pure and Appl. Math. 10(7)(1979), 842-853.
13. POMMERENKE, C., On Starlike and Convex Functions, J. London Math. Soc. 37(1962), 209-224.
14. ROBERTSON, M. S., On the Theory of Univalent Functions, Ann. of Math. 37(1936), 374-409.
15. ROBERTSON, M. S., Univalent Functions $f(z)$ for which $zf'(z)$ is spirallike, Michigan Math. J. 16(1969), 97-101.
16. SCHILD, A., On Starlike Functions of Order α , Amer. J. Math. 87(1965), 65-70.
17. SIZUK, P. I., Regular Functions $f(z)$ for which $zf'(z)$ is θ -spiral Shaped of Order α , Sibirsk. Math. Z 16(1975), 1286-1290, 1371.
18. ŠPÁČEK, L., $\sum_{k=0}^{\infty} \frac{v_k}{k!} \text{Prispevek K Teorii Funkci Prostych}$, Časopis Pěst. Mat. Fys. 62 (1933), 12-19.
19. WIATROWSKI, P., The Coefficients of a Certain Family of Holomorphic Functions, Zeszyty Nauk. Univ. todz. Nauki Mat. Przyrod Ser. II Zeszyt (39) Mat. (1971), 75-85.
20. ZAMORSKI, J. About the Extremal Spiral Schlicht Functions, Ann. Polon. Math. 9 (1962), 265-273.