ANOTHER NOTE ON KEMPISTY’S GENERALIZED CONTINUITY

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ABSTRACT. Under a fairly mild completeness condition on spaces Y and Z we show that every x-continuous function \( f: X \times Y \times Z \to M \) has a "substantial" set \( C(f) \) of points of continuity. Some odds and ends concerning a related earlier result shown by the authors are presented. Further, a generalization of S. Kempisty's ideas of generalized continuity on products of finitely many spaces is offered. As a corollary from the above results, a partial answer to M. Talagrand's problem is provided.

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1. x-CONTINUITY.

The notion of symmetric quasi-continuity introduced by S. Kempisty [1] has been generalized in Lee and Piotrowski [2], to x-continuity. In what follows let \( X, Y, Z \) and \( T \) be spaces. Following Lee and Piotrowski [2] a function \( f: X \times Y \times Z \to T \) is x-continuous if for every \( (p,q,r) \in X \times Y \times Z \), for every neighborhood \( U \times V \times W \) of \( (p,q,r) \) and for every neighborhood \( N \) of \( f(p,q,r) \) there exists a neighborhood \( U' \) of \( p \) with \( U' \subseteq U \) and nonempty open sets \( V' \) and \( W' \) with \( V' \subseteq V \) and \( W' \subseteq W \) such that for all \( (x,y,z) \in U' \times V' \times W' \) it follows that \( f(x,y,z) \in N \).

We shall first show that under certain general assumptions concerning the spaces, x-continuous functions have "large" sets of points of joint continuity. In order to do this we first list some necessary definitions.

Let \( A \) be an open covering of a space \( X \). Then a subset \( S \) of \( X \) is said to be A-small if \( S \) is contained in a member of \( A \). A space \( X \) is called strongly countably complete if there exists a sequence \( \{A_i: i=1,2,\ldots\} \) of open coverings of \( X \) such that and sequence \( \{F_i\} \) of \( A_i \)-small, closed subsets of \( X \) for which \( F_i \supseteq F_{i+1} \) has a non-
empty intersection.

The class of strongly countably complete spaces include countably compact and complete metric spaces. This fact follows easily from a theorem due to A. Arhangel'skiï [3] and Z. Frolik [4] which states that in the class of completely regular spaces, Čech-complete and strongly countably complete spaces coincide (Engelking [5]), see also Frolik [4], where some other properties of these spaces such as their invariance under taking closed, open subspaces or products are discussed.

A space $X$ is called quasi-regular, (Oxtoby [6]) if for every nonempty open set $u$, there is a nonempty open set $V$ such that $clV \subset u$. Obviously, every regular space is quasi-regular.

Let us recall that a function $f: X \times Y \to Z$ is said to be quasi-continuous with respect to $x$, (Kempisty [1], p.188,) if for every $(p,q) \in X \times Y$, fore very neighborhood $N$ of $f(p,q)$ and every neighborhood $U \times V$ of $(p,q)$ there exists a neighborhood $U'$ of $p$ with $U' \subset U$ and a nonempty open set $V' \subset V$ such that for all $(x,y) \in U' \times V'$ we have $f(x,y) \in N$. Quasi-continuity with respect to $y$ can be defined similarly.

**LEMMA 1.** (Lee and Piotrowski [2], Lemma 3 p. 383). Let $X, Y, Z$ and $T$ be spaces and let $F: X \times Y \to Z + T$ be a function. Then $F$ is $x$-continuous if and only if $g: X \times S + T$ is quasi-continuous with respect to $x$, where $S = Y \times Z$ and $g(x,(y,z)) = F(x,y,z)$.

**THEOREM 2.** Let $X$ be a space, $Y$ and $Z$ be spaces such that $Y \times Z$ is quasi-regular, strongly countably complete and let $M$ be metric. If $f: X \times Y \times Z \to M$ is $x$-continuous then for every $x \in X$, the set $C(f)$ of continuity points of $f$ is dense $G_\delta$ subset in $(x) \times Y \times Z$.

**PROOF.** In view of Lemma 1 it is sufficient to prove the following:

**CLAIM.** Let $X$ be a space, $Y$ be a quasi-regular, strongly countably complete and $Z$ be metric. If $f: X \times Y \times Z \to M$ is quasi-continuous with respect to $x$, then for all $x \in X$ the set of points of joint continuity of $f$ is a dense $G_\delta$ subset of $\{x\} \times Y$.

**PROOF.** First we will prove that the set of points of joint continuity of $f$ is dense in $(x) \times Y$. Let $x \in X$, $y \in Y$ and $U \times V$ be any neighborhood $U$ of $x$, contained in $U$, and a nonempty open set $V^1 \subset V$ such that for all $(x',y')$ and $(x'',y'')$ in $U^1 \times V^1$, we have $\rho(f(x',y'), f(x'',y'')) < 1$. Without loss of generality we may assume that $V^1$ is contained in an element $A_1$ of the covering $\mathcal{A}_1$ of $Y$. Let $W^1$ be a nonempty open set such that $cl W^1 \subset V^1$. So $cl W^1$ is $A_1$-small. Then $U^1 \times W^1$ is a neighborhood of $(x,y_1)$, where $y_1 \in W^1$, and since $f$ is quasi-continuous with respect to $x$ at $(x,y_1)$, there is a neighborhood $U^2$ of $x$, contained in $U^1$ and a nonempty open set $V^2 \subset W^1$, such that for all $(x',y')$ and $(x'',y'')$ in $U^2 \times V^2$ we have $\rho(f(x',y'), f(x'',y'')) < 1$. Similarly, we may assume that $V^2$ is contained in an element $A_2$ of the covering $\mathcal{A}_2$. Let $W^2$ be a nonempty open set such that $cl W^2 \subset V^2$. We see, that $cl W^2$ is $A_2$-small.

Now, proceeding by induction we get a neighborhood $U^n \times V^n$ of $(x,y_n)$, $y_n \in V^n$, such that for all $(x',y')$ and $(x'',y'')$ in $U^n \times V^n$, we have $\rho(f(x',y'), f(x'',y'')) < 1/n$ and that $V^n$ is contained in an element $A_n$ of the covering $\mathcal{A}_n$ of $Y$. Moreover, there is a nonempty open sets $V^n$ such that $V^{n+1} \subset cl W^n \subset V^n$. Thus each $cl W^n$ is $A_n$-small, obviously $cl W^n \cap cl W^{n+1} = \emptyset$. Let $n=1$
Then \( y^* \in \bigcap_{n=1}^{\infty} \text{cl } W^n \). Then
\[
(x, y^*) \in \bigcap_{n=1}^{\infty} (U^n \times \text{cl } W^n) \subseteq \bigcap_{n=1}^{\infty} (U^n \times V^n) \subseteq U \times V.
\]

Thus \((x, y^*) \in (U \times V) \cap ([x] \times Y)\) and \((x, y^*)\) is a point of joint continuity of \(f\). This shows the density of the set of points of joint continuity of \(f\) in the set \([x] \times Y\).

The proof that this set is \(G_\delta\) subset of \([x] \times Y\) easily follows, when we recall that the function \(f\) takes values in the metric space \(Z\). This completes the proof of Claim.

Thus, Theorem 2 is shown.

The forthcoming, Proposition 3 is contained in Lemma 5.1 of [6], since any quasi-regular strongly countably complete space is pseudo-complete; take \(B(n) = \) the class of all nonempty open sets that are \(A_n\)-small. Then \(\{B(n)\}\) is a sequence of (pseudo-) bases that shows \(X\) to be pseudo-complete.) We would like to thank the referee who make the above observation.

**Proposition 3.** (Oxtoby [6], Lemma 5.1) Every quasi-regular strongly countably complete space \(X\) is a Baire space.

**Remark 4.** Observe that neither base countability nor metrizability assumptions are made on the considered spaces \(X, Y, Z\) in Theorem 1 while in Theorem 2 of [2] the same conclusion concerning the set of points of continuity is obtained under an extra assumption that \(X\) is first countable, \(Y\) is Baire, \(Z\) is second countable in a neighborhood of any of its points and such that \(Y \times Z\) is Baire.

2. Conditions implying \(x\)-continuity - Counter-examples.

Given spaces \(X\) and \(Y\); a function \(f: X \rightarrow Y\) is said to be quasi-continuous (Martin [8], compare Kempisty [1]) if for every \(x \in X\) and for every neighborhood \(U\) of \(x\) and for every neighborhood \(V\) of \(f(x)\) have: \(U \cap \text{Int } f(V) \neq \emptyset\).

The main result of Lee and Piotrowski [2] is the following:

**Theorem A.** (Lee and Piotrowski [2], Theorem 1, p. 383). Let \(X\) be first countable, \(Y\) be Baire, \(Z\) be second countable such that \(Y \times Z\) is Baire and let \(T\) be regular. If \(f: X \times Y \times Z \rightarrow T\) is:

1. continuous at \(X \times \{y\} \times \{z\}, y \in Y, z \in Z\), and
2. quasi-continuous at points of \(\{x\} \times Y \times \{z\}\) for all \(x \in X\) and \(z \in Z\), and
3. quasi-continuous at points of \(\{x\} \times \{y\} \times Z\) for all \(x \in X\) and \(y \in Y\)

then \(f\) is \(x\)-continuous.

The first natural question which comes up is to check whether the converse of Theorem A is true. Apparently, the following Example 5 settles this question in the negative.

**Example 5.** Let \(f: \mathbb{R}^3 \rightarrow \mathbb{R}\) be defined by
\[
f(x, y, z) = \begin{cases} 
\sin \frac{1}{x^2 + y^2 + z^2}, & \text{if } (x, y, z) \neq (0, 0, 0) \\
0, & \text{otherwise}
\end{cases}
\]
The function $f$ is $x$-continuous, however, fixing $y = 0 = z$ we obtain that $f(x,0,0)$ is not continuous.

Now we shall investigate the necessity of the assumptions in Theorem A, in particular:

(*) - continuity of $f$ at points of $X \times \{y\} \times \{z\}$

(**) - quasi-continuity of $f$ at points of $\{x\} \times Y \times \{z\}$, and

(***) - quasi-continuity of $f$ at points of $\{x\} \times \{y\} \times Z$.

In what follows (Examples 6 and 7) such constructions will be provided.

**EXAMPLE 6.** The assumption (*) is essential. In fact, let us consider a function $f: [-1,1]^3 \to \mathbb{R}$ given as follows

$$f(x,y,z) = \begin{cases} (x,y,z+1), & \text{if } (x,y,z) \in [0,1] \times [0,1] \times [0,1] \\ (x,y,z-1), & \text{if } (x,y,z) \in [-1,0] \times [-1,0] \times [-1,0] \\ (x,y,z), & \text{otherwise} \end{cases}$$

A standard verification that $f$ has the required property (namely $f$ is not $x$-continuous at $(0,0,0)$) is left to the reader. Using somewhat more complex, but still elementary techniques we shall show that also (**) (as well as (***)) is essential. In fact, we have

**EXAMPLE 7.** Consider the function $g:[-1,1]^3 \to \mathbb{R}$ given as follows:

$$g(x,y,z) = \begin{cases} (x,y,z + 1) & \text{if } (x,y,z) \in [-1,1] \times [-1,1] \times [-1,1] \\ (x,y,z), & \text{otherwise} \end{cases}$$

Again, we leave to the interested reader a standard verification that $f$ is not $x$-continuous at $(0,0,0)$.

3. ONE-PROMISING HYPOTHESIS.

Observe that the definition of $x$-continuity at $(p,q,r)$ requires the existence of a "small" neighborhood $U'$ of $p$ and "small" nonempty open sets $V'$ and $W'$ such that $q$ and $r$ "cluster" to $V'$ and $W'$ respectively and such that the set $f(U' \times V' \times W')$ is contained in a "small", previously chosen, open set $N$. This observation prompts us to label this kind of product almost continuity as $1-3$-continuity since we require the existence of only one "small" neighborhood $U'$ (around $p$) of the three neighborhoods $U$, $V$, $W$.

The term "$1-3$-continuity" has been used already, in a different sense in Breckenridge and Nishiura [9].

So, now let us consider "2-3-continuity".

More precisely, given spaces $X$, $Y$, $Z$ and $T$, we say that $f: X \times Y \times Z \to T$ is 2-3-continuous or more specifically $xy$-continuous, if for every $(p,q,r) \in X \times Y \times Z$, for every neighborhood $U \times V \times W$ of $(p,q,r)$ and for every neighborhood $N$ of $f(p,q,r)$ there is a neighborhood $U'$ of $p$, with $U' \subset U$, there is a neighborhood $V'$ of $q$, with $V' \subset V$ and a nonempty open set $W'$, with $W' \subset W$ such that for all $(x,y,z) \in U' \times V' \times W'$ we have $f(x,y,z) \in N$.

Now, 3-3-continuity can be defined easily; the set $W'$ in definition of 2-3-continuity is assumed to be a neighborhood of $r$ - not just only a nonempty open subset of $W$. 
Clearly, every 3-3-continuous (≡ continuous) function is 2-3-continuous; 2-3-continuous functions are 1-3-continuous and the latter are in turn 0-3-continuous (≡ quasi-continuous).

It now follows from a result of T. Neubrunn [10] that if X, Y, Z are "nice" (e.g. Baire, second countable), T-regular then if f: X × Y × Z → T is separately quasi-continuous then it is (jointly) quasi-continuous.

We can present this fact in the following symbolic equality:

"0 + 0 + 0 = 0",

where the numbers (0 or 1) on the left side of the equality stand for quasi-continuity (0) or continuity (1) of the corresponding sections and the numbers on the right (i = 0, 1, 2 or 3) denote the corresponding i-3-continuity of f as a function of three variables.

Theorem A implies that if X, Y, Z and T are as above and if f: X × Y × Z → T is continuous in x and is quasi-continuous in y and is quasi-continuous in z, then f is 1-3-continuous. Consequently, we get:

"1 + 0 + 0 = 1"

In view of the above considerations it is now natural to state the following:

HYPOTHESIS. Let X, Y and Z be Baire, second countable spaces and let T be regular. If f: X × Y × Z → T is:

1) continuous in x, and
2) continuous in y, and
3) quasi-continuous in z,

Then f is 2-3-continuous;

In other words:

"1 + 1 + 0 = 2"

We shall resolve this Hypothesis in the negative in the forthcoming Example 8.

Now we shall exhibit two examples of i-3-continuous functions which are not (1 + 1)-3-continuous, i = 1,2.

EXAMPLE 8. A 1-3-continuous function which is not 2-3-continuous. Let f: \( \mathbb{R}^3 \to \mathbb{R} \) be given by f\((x_1,x_2,x_3)\) = g\((x_1,x_2)\) where g is an arbitrary separately continuous function which is discontinuous at \((0,0)\).

EXAMPLE 9. A 2-3-continuous function which is not 3-3-continuous (≡ continuous). Take f: \( \mathbb{R}^3 \to \mathbb{R} \) to be f\((x_1,x_2,x_3)\) = h\((x_3)\), where h is any function which is continuous except for 0.

Using the above pattern the reader will easily construct 0-3-continuous function (≡ quasi-continuous) which is not 1-3-continuous.

Apparently, the above constructions can be illustrated with the following very specific formula-ready example.

EXAMPLE 10. Let f: \( \mathbb{R}^3 \to \mathbb{R} \) be a function.
\[ f(x_1, x_2, x_3) = g^3_1(x_1, \ldots, x_3), \quad i = 1, 2 \]

where

\[ g^3_1(x_1, \ldots, x_1) = \prod_{j=1}^{3} (x_j)^i, \quad \text{if} \quad \sum_{j=1}^{3} (x_j)^i \neq 0 \]

\[ 0, \quad \text{otherwise} \]

Then \( f \) is \( i \)-3-continuous which is not \( (i + 1) \)-3-continuous, \( i = 1, 2 \).

4. FURTHER GENERALIZATION OF \( i \)-3-CONTINUITY.

Having defined 1-3 and 2-3-continuity for \( f: X_1 \times X_2 \times X_3 \to T \), we shall now extend these ideas to a general case.

Namely, let \( n \) be an arbitrary natural number. We say that \( f: X_1 \times X_2 \times \ldots \times X_n \to T \) is \( A-n \)-continuous if for every \((p_1, p_2, \ldots, p_n) \in X_1 \times X_2 \times \ldots \times X_n\) and for every neighborhood \( U_1 \times U_2 \times \ldots \times U_n \) of \((p_1, p_2, \ldots, p_n)\) and for every neighborhood \( N \) of \( f(p_1, p_2, \ldots, p_n) \) there are neighborhoods \( U_{i,1} \times U_{i,2} \times \ldots \times U_{i,k} \) of the first \( k \) out of \( n \) points \( p_1, p_2, \ldots, p_n \) with \( U_{i,s} \subseteq U_i \) and there are \((n-k)\) nonempty open sets \( V_{i,m} \subseteq U_{i,1} \times \ldots \times U_{i,n-k} \times U_{i,m} \) such that for all \((x_1, x_2, \ldots, x_n) \in U_{i,1} \times \ldots \times U_{i,n-k} \times V_{i,m} \) we have \( f(x_1, x_2, \ldots, x_n) \in N \).

An interested reader will easily observe that the formula

\[ g^n_k(x_1, \ldots, x_k) = \prod_{i=1}^{k} (x_i)^k, \quad \text{if} \quad \sum_{i=1}^{k} (x_i)^k \neq 0 \]

\[ 0, \quad \text{otherwise} \]

where \( f: X^n \to R \) describes a \( k \)-n-continuous function \( f \) given by

\[ f(x_1, \ldots, x_n) = g^n_k(x_1, \ldots, x_k), \quad k = 1, 2, 3, \ldots, n-1. \]

One can also give analogues of Example 8 and 9 for \( k \)-n-continuity.

Studies of \( C(f) \) in hyperspaces for separately continuous functions and related ones were done also in Bögel [11] and Hahn [12].

5. A PARTIAL SOLUTION TO A PROBLEM OF M. TALAGRAND.

M. Talagrand ([13] Problem 3 p. 160) asked whether if \( X \) is Baire, \( Y \) is compact and \( f: X \times Y \to R \) is any separately continuous function, is there the set \( C(f) \) of points of continuity of \( f \) nonempty.

We shall answer this question in the positive if a compact space \( Y \) is additionally first countable.

In fact, we have shown the following result:

**Lemma 12.** (Lee and Piotrowski [2], Lemma 2 p. 381). Let \( X \) be Baire, \( Y \) be first countable and \( Z \) be regular. If \( f: X \times Y \to Z \) is a function such that all its \( x \)-sections \( f_x \) are continuous with the exception of a first category set, and all its \( y \)-sections \( f_y \) are quasi-continuous, then \( f \) is quasi-continuous with respect to \( y \).

It follows from the definition that

**Remark 12.** Every quasi-continuous function with respect to \( y \) is quasi-continuous.

**Lemma 13.** (Marcus [14]). Let \( X \) be a Baire, \( M \) be metric. If \( f: X \to M \) is quasi-continuous, then \( C(f) \), the set of point of continuity of \( f \) is dense \( G_\delta \) subset of \( X \).
PROPOSITION 14. Let X be Baire, Y be compact first countable and let 
f: X \to \mathbb{R} be any separately continuous function. Then C(f) \neq 0.

PROOF. By Lemma 11 and Remark 12 such f is quasi-continuous. Now, since the 
Cartesian product of a compact space and a Baire space is Baire, we are done by 
Lemma 13.

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