TWO DIMENSIONAL LAPLACE TRANSFORMS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. The object of this paper is to obtain new operational relations between the original and the image functions that involve generalized hypergeometric G-functions.

KEY WORDS AND PHRASES: Multidimensional Laplace transforms, Meijer's G-function.

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1. INTRODUCTION.

The integral equation

$$\phi(p, q) = pq \int_0^\infty \int_0^\infty \exp(-px - qy) f(x, y) dy dx, \quad \operatorname{Re}(p, q) > 0 \quad (1.1)$$

represents the classical Laplace transform of two variables and the functions $\phi(p, q)$ and $f(x, y)$ related by (1.1), are said to be operationally related to each other. $\phi(p, q)$ is called the image and $f(x, y)$ the original.

Symbolically we can write

$$\phi(p, q) \overset{..}{=} f(x, y) \quad \text{or} \quad f(x, y) \overset{..}{=} \phi(p, q), \quad (1.2)$$

and the symbol $\overset{..}{=}$ is called "operational".

Meijer's G-function [5] is defined by a Mellin-Barnes type integral

$$G_{u,v}^{m,n}(z \mid \begin{matrix} a_u \\ b_v \end{matrix}) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^u \Gamma(1 - b_j + s) \prod_{j=n+1}^v \Gamma(a_j - s)} z^s ds \quad (1.3)$$

where m, n, u, v are integers with $v > 1$; $0 < n < u$; $0 < m < v$, the parameters a_j and b_j are such that no poles of $\Gamma(b_j - s)$; $j = 1, 2, \dots, m$ coincides with any pole of $\Gamma(1 - a_k + s)$; $k = 1, 2, \dots, n$. Thus $(a_k - b_j)$ is not a positive integer. The path L goes from $-i\infty$ to $+i\infty$ so that all poles of integrand must be simple and those of $\Gamma(b_j - s)$; $j = 1, 2, \dots, m$ lie on one side of the contour L and those of $\Gamma(1 - a_k + s)$; $k = 1, 2, \dots, n$ must lie on the other side. The integrand converges if $u + v < 2(m + n)$ and $|\arg z| < (m + n - \frac{1}{2}u - \frac{1}{2}v)\pi$. For sake of brevity a_u denotes $a_1, a_2, \dots, a_u$.

In the present paper, we propose to establish a couple of formulae for

calculating Laplace transform pairs of two dimensions that involve Meijer's G-function.

2. THE MAIN RESULTS.

- (i) $\bar{\delta} = m + n - \frac{1}{2}(u+v) > 0$, $|\arg \alpha| < \bar{\delta}\pi$,
- (ii) $0 < n < u$, $0 < m < v$, $v > 1$,
- (iii) $\operatorname{Re}(b_j + \xi) > 0$, $j = 1, 2, \dots, m$
- (iv) $\operatorname{Re}(a_k + \xi - \frac{\sigma}{2} - \frac{r}{2}) < 0$, $k = 1, 2, \dots, n$,
- (v) $a_k - b_k$ is not a positive integer, $j = 1, 2, \dots, m$, $k = 1, 2, \dots, n$,
- (vi) r represents the non-negative integer, $0, 1, 2, 3, \dots$, then

$$\begin{aligned} & x^r (xy)^{\sigma-\xi-1} G_{u+2,v}^{m,n+1} \left(\frac{\alpha}{xy} \mid \begin{matrix} 1-\xi, a_u, \sigma+r-\xi \\ b_v \end{matrix} \right) \\ & \quad = p^{-r} (pq)^{\xi-\sigma+1} G_{u+1,v+1}^{m+1,n+1} \left(\alpha \mid \begin{matrix} 1-\xi, a_u \\ \sigma-\xi, b_v \end{matrix} \right) \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} & x^{\delta-r-1} \left(\frac{x}{y} \right)^{\xi-\sigma+1} G_{u+2,v}^{m,n+1} \left(\frac{\alpha x}{y} \mid \begin{matrix} 1-\xi, a_u, \sigma+r-\xi \\ b_v \end{matrix} \right) \\ & \quad = p^{r-\delta-1} \left(\frac{q}{p} \right)^{\xi-\sigma+1} G_{u+3,v+1}^{m+1,n+2} \left(\frac{\alpha q}{p} \mid \begin{matrix} \sigma-\xi+r-\delta, 1-\xi, a_u, \sigma-\xi+r \\ \sigma-\xi, b_v \end{matrix} \right) \end{aligned} \quad (2.2)$$

valid under the conditions:

- (a) $\operatorname{Re}(\delta + \xi - \sigma - r + b_j) > -1$, $j = 1, 2, \dots, m$,
- (b) $\operatorname{Re}(\frac{\delta}{2} + \xi - \sigma - r + a_k) < \frac{1}{4}$, $k = 1, 2, \dots, n$,
- (c) along with (i), (ii), (v) and (vi).

From (2.1) and (2.2), we propose to prove the following relations.

$$\begin{aligned} & x^r (xy)^{\sigma-1} E(b_1, \dots, b_v : a_1, \dots, a_u, \sigma+r : \frac{\alpha}{xy}) \\ & \quad = p^{-r} (pq)^{1-\sigma} E(\sigma, b_1, \dots, b_v : a_1, \dots, a_u : \alpha pq), \end{aligned} \quad (2.3)$$

where $\operatorname{Re}(\sigma) > 0$, $v > u+1$ and r is a positive integer.

$$\begin{aligned} & x^{\delta-r-1} \left(\frac{x}{y} \right)^{1-\sigma} E(\xi+b_1, \dots, \xi+b_v : \xi+a_1, \dots, \xi+a_u, \sigma+r : \frac{x}{y}) \\ & \quad = pq^{r-\delta} E(1+\delta-r, 1+\xi+\delta-\sigma-r+b_1, \dots, 1+\xi+\delta-\sigma-r+b_v : \\ & \quad \quad 2+\delta-\sigma-r, 1+\xi+\delta-\sigma-r+a_1, \dots, 1+\xi+\delta-\sigma-r+a_u, 1+\delta : \frac{q}{p}) \end{aligned} \quad (2.4)$$

valid under the same conditions as (2.3).

The function appearing in (2.3) and (2.4) is MacRobert's E-function, whose

properties are given in [6], pp. 433-434, and [8].

PROOF: The generalized Stieltjes transform of a G-function is given by (see [6], p. 237)

$$\int_0^{\infty} \frac{x^{\xi-1}}{(x+\beta)^{\sigma}} G_{p,q}^{m,n} \left(\alpha x \mid \begin{matrix} a_p \\ b_q \end{matrix} \right) dx = \frac{\beta^{\xi-\sigma}}{\Gamma(\sigma)} G_{p+1,q+1}^{m+1,n+1} \left(\alpha\beta \mid \begin{matrix} 1-\xi, a_p \\ \sigma-\xi, b_q \end{matrix} \right) \quad (2.5)$$

On writing pq for β and multiplying both the sides of (2.5) by $p^{1-r} q$, we have

$$\begin{aligned} \int_0^{\infty} \frac{p^{1-r} q}{(pq+t)^{\sigma}} \cdot t^{\xi-1} G_{u,v}^{m,n} \left(\alpha t \mid \begin{matrix} a_u \\ b_v \end{matrix} \right) dt \\ = \frac{(pq)^{\xi-\sigma+1} p^{-r}}{\Gamma(\sigma)} G_{u+1,v+1}^{m+1,n+1} \left(\alpha pq \mid \begin{matrix} 1-\xi, a_u \\ \sigma-\xi, b_v \end{matrix} \right) \end{aligned} \quad (2.6)$$

Now interpreting with the help of the known result ([4], result (2.83), p. 137), it follows

$$\begin{aligned} \frac{y^{\sigma-1}}{\Gamma(\sigma)} \left(\frac{x}{y} \right)^{\frac{\sigma+r-1}{2}} \int_0^{\infty} t^{\xi-\frac{\sigma}{2}-\frac{r}{2}-\frac{1}{2}} J_{\sigma+r-1}(2\sqrt{txy}) G_{u,v}^{m,n} \left(\alpha t \mid \begin{matrix} a_u \\ b_v \end{matrix} \right) dt \\ = \frac{(pq)^{\xi-\sigma+1}}{\Gamma(\sigma) p^r} G_{u+1,v+1}^{m+1,n+1} \left(\alpha pq \mid \begin{matrix} 1-\xi, a_u \\ \sigma-\xi, b_v \end{matrix} \right). \end{aligned} \quad (2.7)$$

Evaluating the left-hand side integral of (2.7), we get

$$\begin{aligned} x^r (xy)^{\sigma-\xi-1} G_{u+2,v}^{m,n+1} \left(\frac{\alpha}{xy} \mid \begin{matrix} 1-\xi, a_u, \sigma+r-\xi \\ b_v \end{matrix} \right) \\ = p^{-r} (pq)^{\xi-\sigma-1} G_{u+1,v+1}^{m+1,n+1} \left(\alpha pq \mid \begin{matrix} 1-\xi, a_u \\ \sigma-\xi, b_v \end{matrix} \right). \end{aligned} \quad (2.8)$$

The following result will be used in the proof of (2.2).

If $F(p,q) \stackrel{..}{=} f(x,y)$, then

$$x^{\delta-1} f\left(\frac{1}{x}, y\right) \stackrel{..}{=} \int_0^{\infty} \left(\frac{\lambda}{p}\right)^{\frac{\delta}{2}-1} J_{\delta}(2\sqrt{p\lambda}) F(\lambda, q) d\lambda. \quad (2.9)$$

From (2.8) and (2.9), we have

$$\begin{aligned} x^{\delta-r-\sigma+\xi} y^{\sigma-\xi-1} G_{u+2,v}^{m,n+1} \left(\frac{\alpha x}{y} \mid \begin{matrix} 1-\xi, a_u, \sigma+r-\xi \\ b_v \end{matrix} \right) \\ = \frac{q^{\xi-\sigma+1}}{p^{\delta/2-1}} \int_0^{\infty} \lambda^{\frac{\delta}{2}+\xi-\sigma-r} J_{\delta}(2\sqrt{p\lambda}) G_{u+1,v+1}^{m+1,n+1} \left(\alpha q \lambda \mid \begin{matrix} 1-\xi, a_u \\ \sigma-\xi, b_v \end{matrix} \right) d\lambda. \end{aligned} \quad (2.10)$$

On evaluating the right hand side integral and after some simplification, we obtain the desired result (2.2).

In (2.1) and (2.2), reducing the Meijer's G-function to MacRobert's E-function to obtain (2.3) and (2.4).

On specializing the parameters, the G-function can be reduced to MacRobert's E-function, generalized hypergeometric function and other higher transcendental functions. Therefore, the results (2.1) and (2.2) leads to many new operational relations not listed in [4,9,2,3] and other literature.

$$p^{-r}(pq)^{\frac{3}{2}-\sigma} \frac{H^{(1)}_{\sigma-\frac{1}{2}}(\sqrt{pq})}{\sigma-\frac{1}{2}} \frac{H^{(2)}(\sqrt{pq})}{\sigma-\frac{1}{2}} \stackrel{\cdot}{=} 2\pi^{-5/2} \cos\left(\sigma - \frac{1}{2}\right)\pi x^r (xy)^{\sigma-\xi-1} \cdot G_{2,2}^{2,1}\left(\frac{1}{xy} \mid \begin{matrix} 1-\xi, \sigma+r-\xi \\ 1-\sigma-\xi, \frac{1}{2}-\xi \end{matrix}\right) \quad (3.6)$$

$$\begin{aligned}
& p^{-r} (pq)^{\frac{1}{2}-m} W_{k,m}(2i\sqrt{pq}) W_{k,m}(-2i\sqrt{pq}) \\
& \quad = \frac{x^r(xy)^{m+\frac{a}{2}-\frac{1}{2}}}{\sqrt{\pi} \Gamma(\frac{1}{2}-K+m) \Gamma(\frac{1}{2}-K-m)} \\
& \quad \cdot G_{3,3}^{3,1} \left(\frac{1}{xy} \mid \begin{matrix} 1+\frac{a}{2}+K, 1+\frac{a}{2}-K, m+\frac{a}{2}+r+\frac{1}{2} \\ \frac{a+1}{2}-m, \frac{a}{2}+1, \frac{a+1}{2} \end{matrix} \right) \quad (3.7)
\end{aligned}$$

$$\begin{aligned}
& p^{-r} (pq)^{1-\frac{u}{2}-\frac{v}{2}} [e^{i\pi(v-u)/2} H_v^{(1)}(\sqrt{pq}) H_u^{(2)}(\sqrt{pq}) \\
& \quad - e^{i\pi(u-v)/2} H_u^{(1)}(\sqrt{pq}) H_v^{(2)}(\sqrt{pq})] \\
& \quad = 2\pi^{-5/2} (\cos u\pi - \cos v\pi) x^r (xy)^{\frac{1}{2}(u+v+w)-1} \\
& \quad \cdot G_{3,3}^{3,1} \left(\frac{1}{xy} \mid \begin{matrix} \frac{w}{2}, \frac{w+1}{2}, \frac{w+u+v}{2}+r \\ \frac{w-u+v}{2}, \frac{w+u-v}{2}, \frac{w-u-v}{2} \end{matrix} \right) \quad (3.8)
\end{aligned}$$

The following four results are obtained from (2.1) by using Carlson's results [1]:

$$\begin{aligned}
& p^{1-r} \psi(1-2a, 1-2c; 2i\sqrt{pq}) + \psi(1-2a, 1-2c; -2i\sqrt{pq}) \\
& \quad = \frac{2^{2c-2a} x^{r-1} y^{-1}}{\sqrt{\pi} \Gamma(1-2a) \Gamma(1+2c-2a)} G_{3,3}^{2,2} \left(\frac{1}{xy} \mid \begin{matrix} a, a+\frac{1}{2}, r \\ c, c+\frac{1}{2}, \frac{1}{2} \end{matrix} \right) \quad (3.9)
\end{aligned}$$

$$\begin{aligned}
& \sqrt{pq} p^{-r} [\psi(1-2a, 1-2c; 2i\sqrt{pq}) - \psi(1-2a, 1-2c; -2i\sqrt{pq})] \\
& \quad = \frac{-i 2^{2c-2a} x^{r-\frac{1}{2}} y^{-\frac{1}{2}}}{\Gamma(1-2a) \Gamma(1+2c-2a)} G_{3,3}^{2,2} \left(\frac{1}{xy} \mid \begin{matrix} a, a+\frac{1}{2}, r+\frac{1}{2} \\ c, c+\frac{1}{2}, 0 \end{matrix} \right) \quad (3.10)
\end{aligned}$$

$$\begin{aligned}
& p^{-r} (pq)^{1+c} [\exp(2i\sqrt{pq} + ic\pi) \Gamma(-2c, 2i\sqrt{pq}) + \exp(-2i\sqrt{pq} - ic\pi) \Gamma(-2c, -2i\sqrt{pq})] \\
& \quad = \pi^{-\frac{1}{2}} [\Gamma(1+2c)]^{-1} x^{r-1} y^{-1} G_{2,2}^{2,1} \left(\frac{1}{xy} \mid \begin{matrix} 0, r \\ c, c+\frac{1}{2} \end{matrix} \right) \quad (3.11)
\end{aligned}$$

where $\Gamma(a, x)$ denotes incomplete gamma function.

$$\begin{aligned}
& p^{-r} [\psi(2, 2-2c; 2i\sqrt{pq}) - \psi(2, 2-2c; -2i\sqrt{pq})] \\
& \quad = -i\pi^{-\frac{1}{2}} 2^{2c} [\Gamma(1+2c)]^{-1} x^r G_{2,2}^{2,1} \left(\frac{1}{xy} \mid \begin{matrix} 0, 1+r \\ c, c+\frac{1}{2} \end{matrix} \right) \quad (3.12)
\end{aligned}$$

In the last six relations, $G_{3,3}^{3,1}$ or $G_{3,3}^{2,2}$ provides an interpretation for the symbol ${}_3F_2$, as does ψ for ${}_2F_0$. So does $G_{2,2}^{2,1}$ or $G_{2,2}^{1,1}$ for ${}_2F_1$.

3(b). The G-function expressed as a named original function.

The following results are obtained from (2.1) by using the known results [7] on pages 228-234.

$$\begin{aligned}
 & p^{\sigma-2K-1} (pq)^{K-\sigma-\frac{a}{2}} G_{1,5}^{4,1} \left(pq \mid \begin{matrix} \frac{a}{2}-K+1 \\ \sigma-K+\frac{a}{2}, \frac{a+1}{2}+m, \frac{a}{2}+1, \frac{a+1}{2}, \frac{a+1}{2}-m \end{matrix} \right) \\
 & \quad = \frac{\sqrt{\pi}}{\Gamma(2m+1)} \Gamma\left(\frac{1}{2}+K+m\right) x^{2K-\sigma+1} (xy)^{\sigma-K-1} M_{K,m}\left(\frac{2}{\sqrt{xy}}\right) W_{-K,m}\left(\frac{2}{\sqrt{xy}}\right) \quad (3.13)
 \end{aligned}$$

$$\begin{aligned}
 & p^{2K+\sigma-1} (pq)^{1-\sigma-K-\frac{a}{2}} G_{1,5}^{5,1} \left(pq \mid \begin{matrix} \frac{a}{2}+K+1 \\ \sigma+K+\frac{a}{2}, \frac{a+1}{2}+m, \frac{a+1}{2}-m, \frac{a}{2}+1, \frac{a+1}{2} \end{matrix} \right) \\
 & \quad = \sqrt{\pi} \Gamma\left(\frac{1}{2}-K+m\right) \Gamma\left(\frac{1}{2}-K-m\right) x^{1-\sigma-2K} (xy)^{\sigma+K-1} \cdot W_{K,m}\left(\frac{2i}{\sqrt{xy}}\right) W_{K,m}\left(\frac{-2i}{\sqrt{xy}}\right) \quad (3.14)
 \end{aligned}$$

$$\begin{aligned}
 & p^{\sigma-\frac{3}{2}} (pq)^{2-\sigma-\frac{w}{2}} G_{1,5}^{5,1} \left(pq \mid \begin{matrix} \frac{w}{2} \\ \sigma+\frac{w}{2}-1, \frac{w+u+v}{2}, \frac{w-u+v}{2}, \frac{w+u-v}{2}, \frac{w-u-v}{2} \end{matrix} \right) \\
 & \quad = \frac{-i\pi^{5/2}}{2(\cos u\pi - \cos v\pi)} x^{\frac{3}{2}-\sigma} (xy)^{\sigma-2} \cdot [e^{i\pi(v-u)/2} H_v^{(1)}\left(\frac{1}{\sqrt{xy}}\right) H_u^{(2)}\left(\frac{1}{\sqrt{xy}}\right) \\
 & \quad \quad - e^{i\pi(u-v)/2} H_u^{(1)}\left(\frac{1}{\sqrt{xy}}\right) H_v^{(2)}\left(\frac{1}{\sqrt{xy}}\right)] \quad (3.15)
 \end{aligned}$$

$$\begin{aligned}
 & p^{\sigma-\frac{1}{2}} (pq)^{\frac{3}{2}-\frac{w}{2}-\sigma} G_{1,5}^{5,1} \left(pq \mid \begin{matrix} \frac{w+1}{2} \\ \sigma+\frac{w-1}{2}, \frac{w+u+v}{2}, \frac{w-u+v}{2}, \frac{w+u-v}{2}, \frac{w-u-v}{2} \end{matrix} \right) \\
 & \quad = \frac{\frac{5}{2}\pi \frac{1}{2}-\sigma}{2(\cos u\pi + \cos v\pi)} (xy)^{\sigma-\frac{3}{2}} [e^{i\pi(u-v)/2} H_v^{(1)}\left(\frac{1}{\sqrt{xy}}\right) H_u^{(2)}\left(\frac{1}{\sqrt{xy}}\right) \\
 & \quad \quad + e^{i\pi(u-v)/2} H_u^{(1)}\left(\frac{1}{\sqrt{xy}}\right) H_v^{(2)}\left(\frac{1}{\sqrt{xy}}\right)] \quad (3.16)
 \end{aligned}$$

$$\begin{aligned}
 & p^{\sigma-\frac{3}{4}} (pq)^{2-a-\sigma} G_{1,5}^{3,2} \left(pq \mid \begin{matrix} a \\ \sigma+a-b, b, c, 2a-c, 2a-b \end{matrix} \right) \\
 & \quad = 2\sqrt{\pi} x^{\frac{3}{2}-\sigma} (xy)^{\sigma-2} I_{b+c+2a}\left(\frac{1}{\sqrt{xy}}\right) K_{b-c}\left(\frac{1}{\sqrt{xy}}\right) \quad (3.17)
 \end{aligned}$$

$$\begin{aligned}
 & p^{-\sigma} \frac{3}{2} (pq)^{2-\sigma} G_{1,5}^{5,1} \left(pq \mid \begin{matrix} 0 \\ \sigma-1, a, b, -b, -a \end{matrix} \right) \\
 & \quad = \frac{i \pi^{5/2}}{4 \sin a \pi \sin b \pi} x^{\frac{3}{2}-\sigma} (xy)^{\sigma-2} \cdot \left[e^{-i\pi b} H_{a-b}^{(1)}\left(\frac{1}{\sqrt{xy}}\right) H_{a+b}^{(2)}\left(\frac{1}{\sqrt{xy}}\right) \right. \\
 & \quad \quad \left. - e^{i\pi b} H_{a+b}^{(1)}\left(\frac{1}{\sqrt{xy}}\right) H_{a-b}^{(2)}\left(\frac{1}{\sqrt{xy}}\right) \right] \quad (3.18)
 \end{aligned}$$

$$\begin{aligned}
 & p^{-\sigma} \frac{1}{2} (pq)^{\frac{3}{2}-\sigma} G_{1,5}^{5,1} \left(pq \mid \begin{matrix} \frac{1}{2} \\ \sigma-\frac{1}{2}, a, b, -b, -a \end{matrix} \right) \\
 & \quad = \frac{\pi^{5/2} x^{\frac{1}{2}-\sigma}}{4 \cos a \pi \cos b \pi} (xy)^{\sigma-\frac{3}{2}} \cdot \left[e^{-i\pi b} H_{a-b}^{(1)}\left(\frac{1}{\sqrt{xy}}\right) H_{a+b}^{(2)}\left(\frac{1}{\sqrt{xy}}\right) \right. \\
 & \quad \quad \left. - e^{i\pi b} H_{a+b}^{(1)}\left(\frac{1}{\sqrt{xy}}\right) H_{a-b}^{(2)}\left(\frac{1}{\sqrt{xy}}\right) \right] \quad (3.19)
 \end{aligned}$$

$$\begin{aligned}
 & p^{\sigma+2a-1} (pq)^{\frac{3}{2}-a-\sigma} G_{1,5}^{5,1} \left(pq \mid \begin{matrix} \frac{1}{2} + a \\ \sigma+a-\frac{1}{2}, 0, \frac{1}{2}, b, -b \end{matrix} \right) \\
 & \quad = \sqrt{\pi} \Gamma\left(\frac{1}{2} + b - a\right) \Gamma\left(\frac{1}{2} - b - a\right) x^{1-2a-\sigma} (xy)^{\sigma+a-1} W_{a,b}\left(\frac{2i}{\sqrt{xy}}\right) W_{a,b}\left(\frac{-2i}{\sqrt{xy}}\right) \quad (3.20)
 \end{aligned}$$

The following three results are obtained by using Carlson's results [1] on page 239 in (2.1).

$$\begin{aligned}
 & p^{-\sigma} \frac{3}{2} (pq)^{2-\sigma} G_{1,5}^{3,1} \left(pq \mid \begin{matrix} 0 \\ \sigma-1, c, c+\frac{1}{2}, d, d+\frac{1}{2} \end{matrix} \right) \\
 & \quad = \frac{\Gamma(1+2c) \pi^{-\frac{1}{2}} x^{\frac{3}{2}-\sigma}}{\Gamma(1+2c-2d) 2^{2d+1}} (xy)^{\sigma-c-2} \left[e^{ic\pi} {}_1F_1\left(1+2c; 1+2c-2d; \frac{-2i}{\sqrt{xy}}\right) \right. \\
 & \quad \quad \left. + e^{-ic\pi} {}_1F_1\left(1+2c; 1+2c-2d; \frac{2i}{\sqrt{xy}}\right) \right] \quad (3.21)
 \end{aligned}$$

$$\begin{aligned}
 & p^{-\sigma} \frac{3}{2} (pq)^{2-\sigma} G_{1,5}^{5,1} \left(pq \mid \begin{matrix} 0 \\ \sigma-1, c, c+\frac{1}{2}, d, d+\frac{1}{2} \end{matrix} \right) \\
 & \quad = \sqrt{\pi} 2^{-2d} \Gamma(1+2c) \Gamma(1+2d) \cdot x^{\frac{3}{2}-\sigma} (xy)^{\sigma-c-2} \left[e^{ic\pi} \psi\left(1+2c, 1+2c-2d; \frac{2i}{\sqrt{xy}}\right) \right. \\
 & \quad \quad \left. + e^{-ic\pi} \psi\left(1+2c, 1+2c-2d; \frac{-2i}{\sqrt{xy}}\right) \right] \quad (3.22)
 \end{aligned}$$

$$\begin{aligned}
& p^{\frac{5}{2}-\sigma-\delta} \left(\frac{p}{p} \right)^{2-\sigma} G_{3,5}^{3,2} \left(\frac{q}{p} \mid \begin{array}{c} \frac{1}{2}-\delta, 0, \frac{1}{2} \\ \sigma-1, c, c+\frac{1}{2}, d, d+\frac{1}{2} \end{array} \right) \\
& = \frac{\Gamma(1+2c)}{\Gamma(1+2c-2d)} \frac{\pi^{-1/2}}{2^{2d+1}} x^{c+\delta-\frac{1}{2}} y^{\sigma-c-2} \cdot [e^{ic\pi} {}_1F_1(1+2c; 1+2c-2d; -2i\sqrt{\frac{x}{y}}) \\
& \quad + e^{-ic\pi} {}_1F_1(1+2c; 1+2c-2d; 2i\sqrt{\frac{x}{y}})] \quad (3.28)
\end{aligned}$$

$$\begin{aligned}
& p^{\frac{1}{2}-\delta} q^{2-\sigma} G_{3,5}^{5,2} \left(\frac{q}{p} \mid \begin{array}{c} \frac{1}{2}-\delta, 0, \frac{1}{2} \\ \sigma-1, c, c+\frac{1}{2}, d, d+\frac{1}{2} \end{array} \right) \\
& = 2^{-2d} \sqrt{\pi} \frac{\Gamma(1+2c)\Gamma(1+2d)}{\Gamma(1+2c-2d)} x^{\delta+c-\frac{1}{2}} y^{\sigma-c-2} \cdot [e^{ic\pi} \psi(1+2c, 1+2c-2d; 2i\sqrt{\frac{x}{y}}) \\
& \quad + e^{-ic\pi} \psi(1+2c, 1+2c-2d; -2i\sqrt{\frac{x}{y}})] \quad (3.29)
\end{aligned}$$

$$\begin{aligned}
& p^{\frac{3}{2}-\sigma-\delta} \left(\frac{q}{p} \right)^{\frac{3}{2}-\sigma} G_{3,5}^{5,2} \left(\frac{q}{p} \mid \begin{array}{c} -\delta, \frac{1}{2}, 0 \\ \sigma-\frac{1}{2}, c, c+\frac{1}{2}, d, d+\frac{1}{2} \end{array} \right) \\
& = \frac{\sqrt{\pi}i}{2^{2d}} \frac{\Gamma(1+2c)\Gamma(1+2d)}{\Gamma(1+2c-2d)} x^{c+\delta} y^{\sigma-c-\frac{3}{2}} \cdot [e^{ic\pi} \psi(1+2c, 1+2c-2d; 2i\sqrt{\frac{x}{y}}) \\
& \quad - e^{-ic\pi} \psi(1+2c, 1+2c-2d; -2i\sqrt{\frac{x}{y}})] \quad (3.30)
\end{aligned}$$

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