SOME CLASSES OF ALPHA-QUASI-CONVEX FUNCTIONS

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ABSTRACT. Let $C[C, D]$, $-1 \leq D < C \leq 1$ denote the class of functions $g, g(0) = 0, g'(0) = 1$, analytic in the unit disk E such that $\frac{(zg'(z))'}{g'(z)}$ is subordinate to $\frac{1+CZ}{1+DZ}$, $z \in E$. We investigate some classes of Alpha-Quasi-Convex Functions f , with $f(0) = f'(0) - 1 = 0$ for which there exists a $g \in C[C, D]$ such that $(1-\alpha)\frac{f'(z)}{g'(z)} + \alpha\frac{(zf'(z))'}{g'(z)}$ is subordinate to $\frac{1+AZ}{1+BZ}$, $-1 \leq B < A \leq 1$. Integral representation, coefficient bounds are obtained. It is shown that some of these classes are preserved under certain integral operators.

KEY WORDS AND PHRASES. Convex, starlike, quasi-convex, close-to-convex function, Integral representation, Alpha-quasi-convex functions.

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1. INTRODUCTION

Let S, K, S^* and C denote the classes of analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are respectively univalent, close-to-convex, starlike (with respect to the origin) and convex in the unit disc E . In [1], a new subclass C^* of univalent functions was introduced and studied. A function f belongs to C^* if there exists a convex function g such that, for $z \in E$,

$$\operatorname{Re} \frac{(zf'(z))'}{g'(z)} > 0.$$

The functions in C^* are called quasi-convex functions and $C \subset C^* \subset K \subset S$. It is also known that $f \in C^*$, if, and only if, $zf' \in K$. For complete study of C^* , see Noor [2].

A new class Q_α of α -quasi-convex functions has been defined and discussed in some details in [3]. A function f belongs to the class Q_α , α real, if and only if there exists a convex function g such that, for $z \in E$

$$\operatorname{Re} \left[(1-\alpha) \frac{f'(z)}{g'(z)} + \alpha \frac{(zf'(z))'}{g'(z)} \right] > 0 \quad (1.1)$$

We note that $Q_0 = K$ and $Q_1 = C^*$.

In [4], Janowski introduced the class $P[A, B]$. For A and B , $-1 \leq B < A \leq 1$, a function p , analytic in E with $p(0) = 1$ belongs to the class $P[A, B]$, if $p(z)$ is subordinate to $\frac{1+AZ}{1+BZ}$. Also, given C and D , $-1 \leq D < C \leq 1$, $C[C, D]$ and $S^*[C, D]$

denote the classes of functions f analytic in E with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ such that $\frac{(zf'(z))'}{f'(z)} \in P[C, D]$ and $\frac{zf'(z)}{f(z)} \in P[C, D]$ respectively. For $C=1$ and $D=-1$ we note that $C[1, -1] = C$ and $S^*[1, -1] = S^*$. Silvia [5] defines the classes $K[A, B; C, D]$ as follows:

Definition 1.1. A function $f: f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, analytic in E , is said to be in the class $K[A, B; C, D]$, $-1 \leq B < A \leq 1$; $-1 \leq D < C \leq 1$, if there exists a $g \in C[C, D]$ such that $\frac{f'(z)}{g'(z)} \in P[A, B]$.

It is clear that $K[1, -1; 1, -1] = K$ and

$$K[A, B; C, D] \subset K \subset S.$$

We now define the following:

Definition 1.2. Let $\alpha \geq 0$ be real and $f: f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E . Then $f \in Q_{\alpha}[A, B; C, D]$, $-1 \leq B < A \leq 1$; $-1 \leq D < C \leq 1$ if and only if there exists a function $g \in C[C, D]$ such that, for $z \in E$,

$$(1-\alpha) \frac{f'(z)}{g'(z)} + \alpha \frac{(zf'(z))'}{g'(z)} \in P[A, B].$$

It is clear that $Q_{\alpha}[1, -1; 1, -1] = Q_{\alpha}$.

2. MAIN RESULTS

We shall now study some of the basic properties of the class $Q_{\alpha}[A, B; C, D]$. From the definition 1.2, we immediately have:

THEOREM 2.1. Let $F(z) = (1-\alpha)f(z) + \alpha zf'(z)$, where $0 < \alpha < 1$ is real and $z \in E$. Then $f \in Q_{\alpha}[A, B; C, D]$, $-1 \leq B < A \leq 1$; $-1 \leq D < C \leq 1$ if and only if $F \in K[A, B; C, D]$.

We now give the integral representation for the functions in the class $Q_{\alpha}[A, B; C, D]$.

THEOREM 2.2. A function $f \in Q_{\alpha}[A, B; C, D]$, for $\alpha > 0$, $-1 \leq B < A \leq 1$; $-1 \leq D < C \leq 1$, if and only if there exists a function $F \in K[A, B; C, D]$ such that, for $z \in E$,

$$f(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z \frac{1}{\zeta^{\frac{1}{\alpha}-2}} F(\zeta) d\zeta \quad (2.1)$$

PROOF. From (2.1), it follows that

$$\left(\frac{1}{\alpha} - 1\right) z^{\frac{1}{\alpha}-2} f(z) + \alpha z^{\frac{1}{\alpha}-1} f'(z) = z^{\frac{1}{\alpha}-2} F(z),$$

so

$$(1-\alpha)f(z) + \alpha zf'(z) = F(z)$$

and the result follows immediately from theorem 2.1.

THEOREM 2.3. Let $f \in Q_{\alpha}[A, B; C, D]$, $0 < \alpha < 1$ and $-1 \leq B < A \leq 1$; $-1 \leq D < C \leq 1$. Then $f \in K[A, B; C, D]$ and hence is univalent.

PROOF. Silvia [5] has proved that if $f_1 \in K[A, B; C, D]$, then so is

$$F_1(z) = \frac{1+\gamma_1}{\gamma_1} \int_0^z t^{\gamma_1-1} f_1(t) dt, \quad \operatorname{Re} \gamma_1 > 0. \quad (2.2)$$

Using this result and the integral representation (2.2) with $\gamma_1 = \frac{1}{\alpha} - 1$ for $f \in Q_{\alpha}[A, B; C, D]$, we obtain the required result.

For our next theorem, we need the following result due to Silvia [5].

LEMMA 2.1. Let $F \in K[A, B; C, D]$ and $F(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then

$$|b_2| \leq \frac{(C-D) + (A-B)}{2},$$

and

$$|b_3| \leq \begin{cases} \frac{C-D}{6} + \frac{(A-B)(C-D+1)}{3}, & |C-2D| \leq 1 \\ \frac{(C-D)(C-2D)}{6} + \frac{(A-B)(C-D+1)}{3}, & |C-2D| > 1. \end{cases}$$

THEOREM 2.4. Let $F \in Q_{\alpha}[A, B; C, D]$, $0 < \alpha < 1$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.

Then

$$|a_2| \leq \frac{1}{1+\alpha} \left[\frac{(C-D) + (A-B)}{2} \right],$$

and

$$|a_3| \leq \frac{1}{(1+2\alpha)} \begin{cases} \frac{(C-D)}{6} + \frac{(A-B)(C-D+1)}{3}, & |C-2D| \leq 1 \\ \frac{(C-D)(C-2D)}{2} + \frac{(A-B)(C-D+1)}{3}, & |(C-2D)| > 1 \end{cases}$$

PROOF. Since $f \in Q_{\alpha}[A, B; C, D]$, by theorem 2.1, the function

$$F(z) = (1-\alpha)f(z) + \alpha z f'(z)$$

belongs to $K[A, B; C, D]$. Let $F(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

Thus

$$(1-\alpha)\left[z + \sum_{n=2}^{\infty} a_n z^n\right] + \alpha z \left[1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right] = z + \sum_{n=2}^{\infty} b_n z^n$$

or

$$(1-\alpha) \sum_{n=2}^{\infty} a_n z^n + \alpha \sum_{n=2}^{\infty} n a_n z^n = \sum_{n=2}^{\infty} b_n z^n.$$

Equating coefficients of z^n on both sides, we have

$$[(1-\alpha) + \alpha n]a_n = b_n \quad (2.3)$$

Now, using Lemma 2.1 and the relation (2.3), we obtain the required result.

REMARK 2.1. Let $F \in K[A, B; 1, -1]$ and be given by $F(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

Then

$$|b_2| \leq \frac{1}{2} (A-B+2).$$

This result is sharp for the function $F_0 \in K[A, B, 1, -1]$ and defined by

$$F_0(z) = \int_0^z \frac{(1+Aw)}{(1-w)^2(1+Bw)} dw.$$

3. THE CLASS $Q_{\alpha}[1-2\beta, -1; 1-2\gamma, -1]$

In definition 1.2, if we put $A=1-2\beta$, $B=-1$; $C=1-2\gamma$, $D=-1$, then we have the following:

Definition 3.1. A function f , analytic in E , is said to be alpha-quasi-convex of order β type γ , if, and only if, there exists a function

$g \in C[1-2\gamma, -1]$ such that

$$H(\alpha, f) = (1-\alpha) \frac{f'(z)}{g'(z)} + \alpha \frac{(zf'(z))'}{g'(z)} \in P[1-2\beta, -1]$$

REMARK 3.1. Let g be analytic in E . Then $g \in C[1-2\gamma, -1]$ if and only if

$$\operatorname{Re} \frac{(zg'(z))'}{g'(z)} > \gamma, \quad z \in E.$$

Thus $H(\alpha, f) \in P[1-2\beta, -1]$ implies that

$$\operatorname{Re} \left[(1-\alpha) \frac{f'(z)}{g'(z)} + \alpha \frac{(zf'(z))'}{g'(z)} \right] > \beta, \quad z \in E.$$

REMARK 3.2. It follows, from the definition 3.1, that $f \in Q_\alpha[1-2\beta, -1; 1-2\gamma, -1]$ if, and only if $\{(1-\alpha)f + \alpha zf'\} \in K[1-2\beta, -1; 1-2\gamma, -1]$.

We now have the following:

THEOREM 3.1. Let $f \in Q_\alpha[1-2\beta, -1; 1-2\gamma, -1]$ and be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.

Then we have, for $n \geq 2$

$$|a_n| \leq \frac{2(3-2\gamma)(4-2\gamma) \dots (n-2\gamma)[n(1-\beta) + \beta - \gamma]}{n! [1 + \alpha(n-1)]}.$$

This result is sharp and the equality holds for the function f_0 defined as

$$f_0(z) = \begin{cases} \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z \zeta^{\frac{1}{\alpha}-2} (\zeta(1-\gamma)(1-2\beta) + (\beta-\gamma)[1-(1-\zeta)^{2-2\gamma}]) d\zeta, & \gamma \neq 1, \gamma \neq \frac{1}{2} \\ \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z \zeta^{\frac{1}{\alpha}-2} [(1-2\beta) \log(1-\zeta) + \frac{2(1-\beta)\zeta}{1-\zeta}] d\zeta, & \gamma = \frac{1}{2} \\ \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z \zeta^{\frac{1}{\alpha}-2} [2(\beta-1) \log(1-\zeta) + (2\beta-1)\zeta] d\zeta, & \gamma = 1 \end{cases}$$

PROOF. Since $f \in Q_\alpha[1-2\beta, -1; 1-2\gamma, -1]$, the function

$$F(z) = (1-\alpha)f(z) + \alpha zf'(z)$$

belong to $K[1-2\beta, -1; 1-2\gamma, -1]$. Let $F(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

Libera [6] has proved that, for $n \geq 2$,

$$|b_n| \leq \frac{2(3-2\gamma)(4-2\gamma) \dots (n-2\gamma)[n(1-\beta) + \beta - \gamma]}{n!}, \quad (3.1)$$

Now, from relation (2.3), we have

$$a_n = \frac{b_n}{1 + \alpha(n-1)}$$

Using this and (3.1), we obtain the required result

THEOREM 3.2. Let $0 < \lambda \leq 1$ and $0 \leq \beta < 1$. Let f be defined as

$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \zeta^{\frac{1}{\lambda}-2} F(\zeta) d\zeta, \quad \frac{1}{\lambda} \geq 1.$$

and $F \in Q_\alpha[1-2\beta, -1; 1-2\gamma, -1]$ where $0 \leq \lambda \leq 1$, $\alpha \geq 0$. Then $f \in Q_\alpha[1-2\beta, -1; 1-2\gamma, -1]$

PROOF. Let

$$F_1(z) = (1-\alpha)F(z) + \alpha zF'(z), \quad (3.2)$$

and let

$$f_1(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \zeta^{\frac{1}{\lambda}-2} F_1(\zeta) d\zeta. \quad (3.3)$$

Since $F \in Q_\alpha[1-2\beta, -1; 1-2\gamma, -1]$, it follows from remark 3.2 that

$F_1 \in K[1-2\beta, -1; 1-2\gamma, -1]$. We want to show that $f \in Q_\alpha[1-2\beta, -1; 1-2\gamma, -1]$, where $f_1(z) = (1-\alpha)f(z) + \alpha zf'(z)$. Now (3.2) can be written as

$$F_1(z) = (1-\alpha)F(z) + \alpha z F'(z) \\ = \alpha z^{2-\frac{1}{\alpha}} \left(z^{\frac{1}{\alpha}} - 1 \right) F'(z),$$

and using this, we obtain from (3.3)

$$f_1(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \zeta^{2-\frac{1}{\alpha}} \zeta^{\frac{1}{\lambda}} - 2 \left(\zeta^{\frac{1}{\alpha}} - 1 \right) F(\zeta) d\zeta \\ = \frac{\alpha}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \zeta^{\frac{1}{\lambda}} - \frac{1}{\alpha} \left(\zeta^{\frac{1}{\alpha}} - 1 \right) F(\zeta) d\zeta$$

So, integrating by parts,

$$f_1(z) = \frac{\alpha}{\lambda} z^{1-\frac{1}{\lambda}} \left[z^{\frac{1}{\lambda}} - \frac{1}{\alpha} \left(z^{\frac{1}{\alpha}} - 1 \right) F(z) - \int_0^z \left(\frac{1}{\lambda} - \frac{1}{\alpha} \right) \zeta^{\frac{1}{\lambda}-\frac{1}{\alpha}-1} F(\zeta) d\zeta \right] \\ = \frac{\alpha}{\lambda} F(z) + \frac{\alpha}{\lambda} \left(\frac{1}{\alpha} - \frac{1}{\lambda} \right) z^{1-\frac{1}{\lambda}} \int_0^z \zeta^{\frac{1}{\lambda}-2} F(\zeta) d\zeta \\ = \alpha \left[\frac{1}{\lambda} F(z) \right] + \alpha \left[\frac{1}{\lambda} \left(1 - \frac{1}{\lambda} \right) + \frac{1}{\lambda} \left(\frac{1}{\alpha} - 1 \right) \right] z^{1-\frac{1}{\lambda}} \int_0^z \zeta^{\frac{1}{\lambda}-2} F(\zeta) d\zeta \\ = \alpha z \left[\frac{1}{\lambda} z^{-1} F(z) + \frac{1}{\lambda} \left(1 - \frac{1}{\lambda} \right) z^{-\frac{1}{\lambda}} \int_0^z \zeta^{\frac{1}{\lambda}-2} F(\zeta) d\zeta \right] \\ + (1-\alpha) \left[\frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \zeta^{\frac{1}{\lambda}-2} F(\zeta) d\zeta \right]. \\ = \alpha z f'(z) + (1-\alpha) f(z). \quad (3.4)$$

Now in (3.3) $F_1 \in K[1-2\beta, -1; 1-2\gamma, -1]$ and so $f_1 \in K[1-2\beta, -1; 1-2\gamma, -1]$, where we have used (2.2) with $\gamma_1 = \frac{1}{\lambda} - 1, A=1-2\beta, B=-1, C=1-2\gamma$ and $D=-1$. Thus it follows from remark 3.2 and the relation (3.4) that $f \in Q_\alpha[1-2\beta, -1; 1-2\gamma, -1]$, and this completes the proof.

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