NONPARAMETRIC MINIMAL SURFACES IN $\mathbb{R}^3$ WHOSE BOUNDARIES HAVE A JUMP DISCONTINUITY

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(Received January 21, 1987 and in revised form February 18, 1987)

ABSTRACT. Let $\Omega$ be a domain in $\mathbb{R}^2$ which is locally convex at each point of its boundary except possibly one, say $(0,0)$, $\phi$ be continuous on $\partial \Omega \setminus \{(0,0)\}$ with a jump discontinuity at $(0,0)$ and $f$ be the unique variational solution of the minimal surface equation with boundary values $\phi$. Then the radial limits of $f$ at $(0,0)$ from all directions in $\Omega$ exist. If the radial limits all lie between the lower and upper limits of $\phi$ at $(0,0)$, then the radial limits of $f$ are weakly monotonic; if not, they are weakly increasing and then decreasing (or the reverse). Additionally, their behavior near the extreme directions is examined and a conjecture of the author's is proven.

KEYS WORDS AND PHRASES. Dirichlet problem, Variational Solution and Nonparametric Minimal surfaces.


1. INTRODUCTION.

How does the generalized solution of the Dirichlet problem for the minimal surface equation with boundary values $\phi$ behave when $\phi$ has a jump discontinuity (say at the origin)? Under certain mild conditions on the domain $\Omega \subseteq \mathbb{R}^2$, we shall show that the radial limits at $(0,0)$ of the solution, denoted $Rf(\theta)$, exist for all $\theta \in (a,\beta)$, where $\Omega = \{(r,\theta) | a < \theta < \beta, 0 < r < r(\theta)\}$. Further, on at most three intervals (i.e. $[a,\alpha^-]$, $[\theta_L, \theta_R]$, $[\beta^-, \beta]$) $Rf(\theta)$ is constant and elsewhere it is strictly monotonic.

If $Rf(\theta)$ lies between the lower and upper limits of $\phi$ at $(0,0)$, then $Rf$ is weakly monotonic on $[a,\alpha^-]$. If not, then $Rf$ is not monotonic on $[a,\beta]$ but it is weakly monotonic on $[\alpha^- a + \pi]$ and on $[\beta - \pi, \beta]$. Under some smoothness and non-tangency assumptions, we shall show that $\alpha^- a + \pi$ or $\alpha^- a + \pi$ and $\beta - \beta - \pi$. We shall also show that $\theta_L = \theta_L + \pi$ when $\theta_L$ and $\theta_R$ occur. Thus there is at most one interval on which $Rf(\theta)$ is constant.

2. PRELIMINARIES.

By $\Omega$ we will mean a bounded open subset of $\mathbb{R}^2$ with the following properties:
(a) $\Omega$ is connected and simply connected. (b) $\partial \Omega$ is Lipschitz and $N = (0,0) \in \partial \Omega$. (c) $\Omega$ is locally convex at each point of its boundary except possibly $N$. (d) In
polar coordinates \((r, \theta)\) about \(N, \Omega = \{(r, \theta) \mid \alpha < \theta < \beta, 0 < r < r(\theta)\}\) with \(-\pi < \alpha < 0 < \beta < \pi\).

From (d) we see that near \(N\), the \(x\)-axis divides \(\partial \Omega\) into two components.

**DEFINITION.** Let \(\Omega\) be as above. We will denote by \(C^*(\partial \Omega)\) those functions \(\phi \in C^0(\partial \Omega/(N))\) such that

\[
\phi(N^+) = \lim \phi(P) \text{ as } P \in \partial \Omega \cap \{(x,y) \mid y > 0\} \text{ approaches } N \text{ and }
\phi(N^-) = \lim \phi(P) \text{ as } P \in \partial \Omega \cap \{(x,y) \mid y < 0\} \text{ approaches } N
\]

each exist.

Notice \(\phi \in C^*(\partial \Omega)\) implies \(\phi\) has a jump discontinuity at \(N\) (possibly with jump 0).

**DEFINITION.** Let \(\phi \in C^*(\partial \Omega)\). Define \(f = f(\cdot, \phi)\) to be the function in \(BV(\Omega)\) which minimizes

\[
J(v) = \int_{\Omega} |Dv|^2 + \int_{\partial \Omega} \sqrt{v(v - \phi)}
\]

for \(v \in BV(\Omega)\).

Notice \(f \in C^2(\Omega) \cap C^0(\partial \Omega/(N))\) and \(f = \phi\) on \(\partial \Omega/(N)\).

We set

\[
S_0 = S_0(\phi) = \{(x,y,f(x,y)) \mid (x,y) \in \Omega\}
\]

and

\[
\Gamma_0 = \Gamma_0(\phi) = \{(x,y,\phi(x,y)) \mid N \neq (x,y) \in \partial \Omega\}.
\]

Let \(S\) be the closure of \(S_0\), \(\Gamma\) be the closure of \(\Gamma_0\), \(\Gamma^+\) be the closure of \(\Gamma \cap \{(x,y) \mid y > 0\}\), and \(\Gamma^-\) be the closure of \(\Gamma \cap \{(x,y) \mid y < 0\}\).

Throughout this paper, we will make the following

**ASSUMPTION.** \(f \notin C^0(\partial \Omega)\).

We will need to represent \(S\) parametrically. Let us set \(E = \{(u,v) \mid u^2 + v^2 < 1\}\), \(B = \{(u,v) \in E \mid v > 0\}\), \(\partial^+B = \{(u,v) \in \partial E \mid v > 0\}\), \(\partial^-B = \{(u,0) \mid -1 < u < 1\}\), \(B^+ = B \cup \partial^+B\), and \(B^- = B \cup \partial^-B\). Using the methods of [1] or [2], we can prove the following propositions.

**PROPOSITION 1.** There exists \(X = (z,y,z) \in C^0(\overline{B}; \mathbb{R}^3) \cap C^2(B; \mathbb{R}^3)\) such that \(X\) maps \(B\) homeomorphically onto \(S_0\), \(X\) maps \(\partial^+B\) strictly monotonically onto \(\Gamma_0\), \(X\) maps \(\partial^-B\) into the \(z\) axis, \(X(-1,0) = (0,0,\phi(N^-))\), \(X(1,0) = (0,0,\phi(N^+))\), and

\[
X_u \cdot X_v = 0
\]

\[
X_u^2 = X_v^2
\]

\[
X_{uu} + X_{vv} = 0
\]

on \(B\). Also, \(X\) extends across \(\partial^-B\) by reflection to a function in \(C^2(E; \mathbb{R}^3)\) and

\[
X_u(u,0) = (0,0,z_u(u,0))
\]

\[
X_v(u,0) = (x_v(u,0),y_v(u,0),0)
\]

for \(-1 < u < 1\).
For each \( \alpha < \theta < \beta \) and \( t > 0 \), define

\[
\lambda(t, \theta) = (t \cos(\theta), t \sin(\theta)),
\]

\[
\omega(t, \theta, \phi) = X^{-1}(\lambda(t, \theta), f(\lambda(t, \theta))),
\]

\[
\lim_{t \to 0^+} Rf(\theta) = t \to 0^+ f(\lambda(t, \theta)) \text{ if this exists}.
\]

Set \( Rf(\alpha) = \phi(N^-) \), \( Rf(\beta) = \phi(N^+) \), \( u(\alpha) = -1 \), and \( u(\beta) = 1 \).

**Proposition 2.** For all \( \alpha < \theta < \beta \), there is a unique \( u(\theta) \in [-1, 1] \) such that

\[
u(t, \theta) \to (u(\theta), 0) \text{ as } t \to 0^+
\]

and

\[
Rf(\theta) = z(u(\theta), 0).
\]

Further, \( u(\cdot) \in C^0([\alpha, \beta]) \), \( Rf \in C^0([\alpha, \beta]) \), and

\[
X_v(u(\theta), 0) = |z_u(u(\theta), 0)| (\cos(\theta), \sin(\theta), 0)
\]

for all \( \theta \in (\alpha, \beta) \) with \( |u(\theta)| < 1 \).

**Remark.** If \( X \) has no branch points on \( \{(u(\theta), 0) \mid \theta_1 < \theta < \theta_2\} \), then \( u(\cdot) \) is strictly increasing on \( [\theta_1, \theta_2] \). Also, \( u(\cdot) \) is weakly increasing on \( [\alpha, \beta] \).

From the proof of Theorem 3.2 of [1], we have the following

**Lemma 1.** Suppose \( \alpha < \theta_1 < \theta_2 \leq \beta \) and \( \theta_2 - \theta_1 \leq \pi \). Then \( Rf \) is weakly monotonic on \( [\theta_1, \theta_2] \). Further, \( X \) maps \( \{(u, 0) \mid u(\theta) \leq u \leq u(\theta_2)\} \) strictly monotonically into the \( z \)-axis.

3. **Boundary Behavior.**

**Definition.** We will say condition \( * \) holds (for \( \phi \in C^*(\partial\Omega) \)) if \( Rf(\theta) \equiv Rf(\theta, \phi) \) lies between \( \phi(N^-) \) and \( \phi(N^+) \) whenever \( \alpha < \theta < \beta \).

**Remark.** If \( \beta - \alpha \leq \pi \), it follows from Lemma 1 or from standard barrier arguments that \( * \) holds for all \( \phi \in C^*(\partial\Omega) \).

**Theorem 1.** Suppose \( * \) holds. Then

- \( X \) is strictly monotonic on \( \partial^- B \), \( Rf \) is weakly monotonic on \( [\alpha, \beta] \), \( S \) has no branch points in \( E \), \( Rf \) is constant on \( [\alpha, \alpha'] \)
- and \( [\beta', \beta] \), and \( Rf \) is strictly monotonic on \( [\alpha', \beta'] \), for some \( \alpha', \beta' \in [\alpha, \beta] \) with \( \alpha' < \beta' \).

Suppose \( * \) does not hold. Then

- \( X \) has one branch point, \( (u(0), 0) \), in \( E \), \( z(\cdot, 0) \) is strictly increasing (decreasing) on \( [-1, u(0)] \) and strictly decreasing (increasing) on \( [u(0), 1] \), \( Rf \) is constant on
- \( (i) [\alpha, \alpha'] \), \( [\beta, \theta_1] \), and \( [\beta', \beta] \), \( Rf \) is strictly increasing (decreasing) on \( [\alpha', \theta_1] \) and \( Rf \) is strictly decreasing (increasing) on \( [\theta_1, \beta'] \), for some \( \alpha', \beta', \theta_1 \), \( \theta_1 \in [\alpha, \beta] \) with \( \alpha' < \theta_1 \) and \( \theta_1 + \pi \leq \theta < \beta' \).

**Proof.** From Lemma 1, we see that \( * \) holds iff \( Rf \) is weakly monotonic on \( [\alpha, \beta] \) and if \( * \) fails to hold, then \( Rf \) is weakly monotonic on \( [\alpha, \alpha + \pi] \) and on \( [\beta - \pi, \beta] \).

From [3] we know that \( X \) is strictly monotonic on a subset of \( \partial\Omega \) iff it is weakly monotonic there. Since \( X(u(\theta), 0) = (0, 0, Rf(\theta)) \), \( X \) has at most one branch point in
E, which can only occur at \((u(0),0)\) \cite{14}). Using Proposition 2 and the subsequent remark, we see either that one of the conclusions of Theorem 1 holds or that \(X\) is monotonic on \(\mathbb{B}\) and has a branch point at \((u(0),0)\). We will eliminate this possibility.

In the case to be eliminated, \(Rf\) is weakly monotonic (say increasing) on \([a,b]\), strictly increasing on \([a',a]\), constant on \([a,a']\), and strictly increasing on \([a',b]\), for some \(a < a' < b\). We may rotate the \(x\)-\(y\) plane so that \(\theta_R = 0\) and (by a conformal map of \(B\) into \(B\) fixing \((-1,0)\) and \((1,0)\)) we may assume that \(u(0) = 0\). As in \cite{5}, there exist neighborhoods \(U\) and \(U'\) of \(0\) in \(E\) and a \(C^1\)-diffeomorphism \(F: U' \to U\) with \(DF(0) = e^{-i\theta}d\) for some \(\theta \neq 0 \in \mathbb{R}\) such that

\[
(z + ix)^m = (F(w))^m
\]

for all \(w \in U'\), where \(0 < \theta < a + 1b\) and \(n > m > 1\) are integers. Suppose we set \(w = s + it = F(w)\) and \(x = x \cdot F^{-1}, y = y \cdot F^{-1}, z = z \cdot F^{-1}\). Then \((z + ix)(w) = \omega^m\) for \(w \in U\). Let \(y\) be the image of the real axis under \(F\). Then \(y\) is tangent to the real axis at the origin and, since \(x(w) = 0\) for \(w\) real, \(x(w) = 0\) for \(w \in U\). If \(\omega = re^{i\delta}\), then \(x(r,\delta) = r^m \sin(m\delta)\) and the only curves on which \(x\) vanishes are \(\delta = k\pi/m\) for all integers \(k\). Thus \(y\) must be the real axis in \(U\). Since \(y(w) = 0\) for \(w\) real, \(y(w) = 0\) for \(w\) real. This means that \(b = 0\) and \(y(\omega) = a \operatorname{Im}(\omega^n) + o(|\omega|^n)\). If \(\sigma\) is a curve in \(U\) from \((r,\delta) = (\epsilon,0)\) to \((r,\delta) = (\epsilon,\pi)\) (\(\epsilon\) small) such that \((x(\sigma), y(\sigma))\) is star-shaped with respect to the origin, then the sign pattern of \(x(\sigma)\) is ++--- and \(y(\sigma)\) is ++---+. Thus \(m\) must be 2, \(n\) must be 3, \(z(s,o) = s^2\), and so \(Rf(\theta) = z(F(u(\theta)))\) cannot be monotonic on \((\alpha,\beta)\). Q.E.D.

In \cite{1}, the case \(\phi \in C^0(\partial B)\) and \(\beta - \alpha > \pi\) is considered and the conjecture that \(\theta_R - \theta_L = \pi\) is mentioned. The following theorem proves that this is always true.

**THEOREM 2.** In case (ii) of Theorem 1, \(\theta_R - \theta_L = \pi\).

**PROOF.** If \(Q\) is an interior branch point of \(X\), then there is a unique unit vector \(n(Q)\) such that as \(P \in E\) approaches \(Q\), the unit normal \(n(P)\) to \(X(E)\) at \(P\) approaches \(n(Q)\) \cite{6}. Since

\[
X_u(u(\theta),0) = (0,0,z_u(u(\theta),0))
\]

and

\[
X_v(u(\theta),0) = |z_u(u(\theta),0)|(\cos(\theta), \sin(\theta),0),
\]

we see that \(n(\theta) = n(u(\theta),0)\) is \(\mp (\sin(\theta), -\cos(\theta), 0)\) when \(\alpha < \theta < \theta_L\) or \(\theta_R < \theta < \beta\). If we let \(\theta \to \theta_L+\), we get \(n(Q) = \mp (\sin(\theta_L), -\cos(\theta_L), 0)\) and if we let \(\theta \to \theta_R+\), we get \(n(Q) = \mp (\sin(\theta_R), -\cos(\theta_R), 0)\) where \(Q = (u(0),0)\). Thus \(\theta_R = \theta_L + \pi\). Q.E.D.

A question of interest is to determine the asymptotic behavior of \(Rf(\theta)\) for \(\theta > \theta_R\) near \(\theta_R\). A discussion of the asymptotic behavior of \(Rf(\theta)\) for \(\theta < \theta_L\) near \(\theta_L\) is similar. We may assume that \(Rf\) is increasing on \([\theta_R, \beta]\).

As in the proof of Theorem 1, let us assume that \(\theta_R = 0\) and \(u(0) = 0\); then \(Rf(\theta) = z(F(u(\theta)))\) and \(z(s,0) = s^2\). Since \(z(w) + ix(w) = \omega^2, \omega = (z + ix)^{1/2}\) and \(y = a \operatorname{Im}(z + ix)^{3/2} + o(|z|+ix|^{3/2})\). Thus
When \( x = 0 \), we get

\[
y'_{\chi}(z) = 3/2 \ a \ z^{1/2} + o(|z|^{1/2}).
\]

Next, if \( 0 = \theta_{R} < \theta < \beta' \), then \( Rf(\theta) \) is equal to that value of \( z > 0 \) for which \( y'_{\chi}(z) = \tan(\theta) \). For this value of \( z \),

\[
z + o(|z|) = \left( \frac{2 \tan(\theta)}{3a} \right)^2
\]

and so asymptotically as \( \theta \to 0^+ \),

\[
Rf(\theta) \sim \left( \frac{2}{3a} \right)^2 \theta^2.
\]

We wish to examine the behavior of \( Rf(\theta) \) near \( \theta = \alpha \) and \( \theta = \beta \).

**Theorem 3.** Let \( \phi \in C^\infty(\Omega) \) and let \( f \in BV(\Omega) \) minimize \( J(\cdot, \phi) \) over \( BV(\Omega) \).

Suppose that \( \Gamma^+ (\Gamma^-) \) is a \( C^1 \) curve in a neighborhood of \((N, \phi(N^+)) ((N, \phi(N^-)))\) which meets the \( z \)-axis nontangentially. Suppose further that the unit normal to the graph of \( f \) extends continuously to the corner formed by \( \Gamma^+ (\Gamma^-) \) and the \( z \)-axis. Then \( \beta^- = \beta \) or \( \beta^- = \beta - \pi \) (\( \alpha^- = \alpha \) or \( \alpha^- = \alpha + \pi \)).

**Proof.** The proof is essentially the same as that of Theorem 2. We will prove \( \beta^- = \beta \) or \( \beta^- = \beta - \pi \). Let \( \theta < \beta^- \) approach \( \beta^- \); then \( n(\theta) \) approaches \( n(\beta^-) = \pm (\sin(\beta^-), -\cos(\beta^-), 0) \). Since the normal to the corner is \( \pm (\sin(\beta), -\cos(\beta), 0) \), we see that \( \beta^- = \beta \) or \( \beta^- = \beta - \pi \).

**Remark.** If \( \Gamma^+ (\Gamma^-) \) is a line segment in a neighborhood of \((N, \phi(N^+)) (N, \phi(N^-))\) which meets the \( z \)-axis nontangentially, then [7] (also [9]) implies that the hypotheses of Theorem 3 are satisfied.

Let us say that a "fan" exists at \( \theta_0 \) when \( Rf(\theta) \) is constant on a nontrivial interval containing \( \theta_0 \). Since \( \beta - \alpha < 2\pi \), we get

**Corollary.** Suppose that the hypotheses of Theorem 3 are satisfied for \( \Gamma^+ \) and \( \Gamma^- \). Then no more than one "fan" can occur.

**4. Examples.**

**Example 1.** (the helicoid). Consider the functions \( f(x,y) \) over \( \hat{\Omega} = \{(r, \theta) | 0 < \theta < \beta, 0 < r < 1 \} \) with \(-\pi < \alpha < \beta < \pi \) whose graph is given parametrically by

\[
Y(s,t) = (t \cos(s), t \sin(s), s).
\]

Then \( \phi = f \in C^\infty(\hat{\Omega}) \), \( Rf(\theta) = \theta \), and \( \Gamma^+ \) meet the \( z \)-axis at right angles. Here we see that \( Rf \) is strictly increasing, \( \alpha^- = \alpha \), and \( \beta^- = \beta \).

**Example 2.** (Scherk's surface). Consider

\[
f(x,y) = \ln(\sin(y)) - \ln(\sin(x))
\]

over \( \Omega = \{(r, \theta) | 0 < r < 1, \alpha < \theta < \beta \} \), where \( 0 < \alpha < \beta < \pi/2 \).

Then \( Rf(\theta) = \ln(\tan(\theta)) \) and \( \Gamma^+ \) meet the \( z \)-axis at right angles. Notice \( \Gamma^+ \) is strictly increasing on \([\alpha, \beta]\), \( \alpha^- = \alpha \), and \( \beta^- = \beta \).

**Example 3.** Here we have an example in which \( \Omega \) is convex and \( \alpha^- \neq \alpha \). Let \( \hat{\Omega}^- = \{(r, \theta) | -3\pi/4 < \theta < 3\pi/4, 0 < r < 1 \} \), \( \phi \in C^0(\hat{\Omega}^-) \) be zero on \( r = 1, -3\pi/4 < \theta < 3\pi/4 \) and \( 0 = 1 - r \) on \( \theta = \pm 3\pi/4, 0 \leq r \leq 1 \), and \( f \in C^2(\hat{\Omega}^-) \cap C^0(\hat{\Omega}^-)(\bar{\Omega}) \) be the variational solution of the Dirichlet problem (for
the minimal surface equation) in \( \Omega \) with boundary data \( \phi \). Next let \( 0 < \varepsilon < \pi/4 \) and define \( \Omega = \{ (r, \theta) | r - \pi/2 < \theta < \pi + \pi/2, 0 < r < 1 \} \). If we set \( \phi = f \) on \( \partial \Omega \), then \( \phi \in C^*(\partial \Omega) \) and \( f \) minimizes \( J \). Notice \( \alpha = -\pi/2 + \varepsilon, \beta = \pi/2 + \varepsilon, \alpha' = \pi/2, \) and \( \beta' = \beta \). Also \( \Gamma^- \) meets the \( z \)-axis tangentially.

**EXAMPLE 4.** (See the discussion of this example in [8].) Let \( \xi \in (\pi/2, \pi) \). Set \( A = (0,0,1), B = (\sin(\xi),0,\cos(\xi), C = (\sin(2\xi), 0,\cos(2\xi)), D = (0,1,0), \) \( \xi = (0,-1,0) \) and \( M = (0,0,0) \). Consider the quadrilateral \( Q_1 \) with successive vertices \( B,D,C,M \) and let \( S_1 \) be the surface of least area spanning \( Q_1 \). Since \( Q_1 \) has a convex injective projection on the \( x-y \) plane, \( S_1 \) is the graph of a function \( g(x,y) \) over the \( x-y \) plane. Now extend \( S_1 \) by reflection across the line segment \( BM \) to a surface \( S \); the boundary of \( S \) is the polygon \( \Gamma \) with successive vertices \( A,E,B,D,C,M \). Let \( \Omega \) be the open subset of the \( x-y \) plane bounded by the projection of \( \Gamma \) on the \( x-y \) plane; notice \( \alpha = -\pi \) and \( \beta = \pi/2 \). Using Theorem 1, we see that \( S_0 = S/\Gamma \) is the graph of a function \( f(x,y) \) over \( \Omega \). Notice \( Rf(\theta) \) is 0 if \( -\pi \leq \theta \leq 0 \), \( Rf(\theta) \) is increasing on \([0,\pi/2]\) (by Theorem 1 (i) and the Corollary to Theorem 3), and \( \Gamma^- \) makes an angle of \( 2(\pi-\varepsilon) \) with the positive \( z \)-axis.

This last part shows that for any angle \( \delta \in (0,\pi) \), we can set \( \xi = \pi-\delta/2 \) and find an example in which \( \alpha' = \alpha + \pi, Rf(\theta) \) is (weakly) increasing on \([\alpha,\beta]\), and \( \Gamma^- \) intersects the positive \( z \)-axis in an angle of \( \delta \).

**REMARK.** In [2], the behavior of a (nonparametric) solution of an equation of prescribed mean curvature with prescribed boundary values in a domain with a reentrant corner is examined. The results of [2] can be extended to the case in which \( \phi \) has a jump discontinuity. In fact, by combining the work in [2] with the techniques used above, Theorems 1, 2, and 3 and the Corollary can be proven in this new situation.

**ACKNOWLEDGEMENT.** I wish to thank Professor Alan Elcrat and especially Professor Robert Gulliver for their encouragement, useful suggestions, and incisive questions. I also wish to thank my wife, Sherry, for her assistance.

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