COMPUTATION OF RELATIVE INTEGRAL BASES FOR ALGEBRAIC NUMBER FIELDS

MAHMOOD HAGHIGHI

Department of Computer Science
Bradley University
Peoria, IL 61625

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ABSTRACT. At first we are given conditions for existence of relative integral bases for extension $(K;k) = n$. Then we will construct relative integral bases for extensions $O_{K_6}({}^{6\sqrt{-3}})/O_{k_2}({}^{\sqrt{-3}})$, $O_{K_6}({}^{6\sqrt{-3}})/O_{k_3}({}^{3\sqrt{-3}})$, $O_{K_6}({}^{6\sqrt{-3}})/\mathbb{Z}$.

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1. EXISTENCE OF A RELATIVE INTEGRAL BASES.

The following criterion has been shown in [1] for existence of a Relative Integral Bases, for any finite extension K/k .

THEOREM 1.1. Let $(K;k) = n$, and let h_k be an odd integer, then O_K has a "relative integral bases" over $O_k \leftrightarrow d_{K/k}$ is a principal ideal. See also [2].

COROLLARY 1.2. If $O_K = \text{P.I.D.}$, then $h_k = 1$ and $d_{K/k} = \text{P.I.}$ Therefore for every finite extension of k where $O_k = \text{P.I.D.}$, a relative integral bases exists.

Let $k_1 = \mathbb{Q}$, $k_2 = \mathbb{Q}({}^{\sqrt{-3}})$, $k_3 = \mathbb{Q}({}^{3\sqrt{-3}})$, $K_6 = \mathbb{Q}({}^{6\sqrt{-3}})$. Since $h_{k_1} = h_{k_2} = h_{k_3} = 1$, so O_{K_1} , O_{K_2} , O_{K_3} are P.I.D. and then by corollary 1.2, relative integral bases for extensions K_6/k_1 , K_6/k_2 , K_6/k_3 exists.

Now, we will compute the relative discriminant for the extensions. Let $(K;k)=n$ and for some $\theta \in K$, $O_K = O_k(\theta)$ and θ satisfies an equation $F(\theta) = 0$ of degree n . Then $D_{K/k} = (F'(\theta) = \prod(\theta - \theta^{(t)}))$, where $\theta, \theta^{(1)}, \theta^{(2)}, \dots, \theta^{(n)}$ are conjugates [3].

Since extensions K_2/K_1 , K_3/K_1 have discriminant divisible by 3 [3], by theorem in [3] discriminants K_6/k_2 , K_6/k_3 , K_6/k_1 are also divisible by 3 and 3 is completely ramified in k_1, k_2, k_3 .

For extension K_6/k_2 , $\theta = {}^{6\sqrt{-3}}$ we therefore have:

$$D_{K_6/k_2} = (\theta - \theta^{(1)})(\theta - \theta^{(2)}) = ({}^{6\sqrt{-3}} - \rho {}^{6\sqrt{-3}}) ({}^{6\sqrt{-3}} - \rho^2 {}^{6\sqrt{-3}}),$$

$$D_{K_6/k_2} = (-3)^{4/3} \text{ for } \rho = \frac{-1 + \sqrt{-3}}{2}. \text{ By the definition in [4],}$$

$$d_{K_6/k_2} = N_{K_6/k_2}(D_{K_6/k_2}) = (-3)^4.$$

For extension K_6/k_3 , $\theta = \sqrt[6]{-3}$, $D_{K_6/k_3} = (\theta - \theta^{(1)}) = (-3)^{1/6}$, then $d_{K_6/k_3} = (-3)^{1/2}$.

By theorem in [4], $D_{K_6/k_1} = D_{K_6/k_2} \cdot D_{K_2/k_1} = (-3)^{4/3} \cdot (-3)^{1/2} = (-3)^{11/6}$, then $d_{K_6/k_1} = (-3)^{11}$.

Now we will construct relative integral bases for the extensions. See also [5] for associated work.

For K_3/k_1 , $O_{K_3} = (1, \sqrt[3]{-3}, \sqrt[3]{(-3)^2}) \cdot Z$, [3].

For K_2/k_1 , $O_{K_2} = (1, \frac{1 + \sqrt{-3}}{2}) \cdot Z$, [3].

2. RELATIVE INTEGRAL BASES FOR $O_6(\sqrt[6]{-3})/O_2(\sqrt{-3})$.

Let $O_6 = (1, \alpha, \beta)O_2$ for α, β in O_6 . By theorem in [6], $\text{disc}(1, \alpha, \beta) = d_{K_6/k_2}$,

$$\text{disc}(1, \alpha, \beta) = \begin{vmatrix} 1 & \alpha & \beta \\ 1 & \rho\alpha & \rho^2\beta \\ 1 & \rho^2\alpha & \rho\beta \end{vmatrix} = d_{K_6/k_2} = (-3)^4.$$

Now $\alpha^2\beta^2(3\rho^2 - 3\rho)^2 = (-3)^4$ and from here $\alpha \cdot \beta = \sqrt{-3}$.

We may take $\alpha = \sqrt[6]{-3}$ and $\beta = \sqrt[6]{(-3)^2}$, because they satisfy an $\alpha \cdot \beta = \sqrt{-3}$ and they are in O_6 .

Since $N_{6/3}(\alpha) = \sqrt[3]{-3}$ and $N_{6/3}(\beta) = \sqrt[3]{(-3)^2}$ are in O_3 , we have:

$$O_6 = (1, \sqrt[6]{-3}, \sqrt[6]{(-3)^2}) O_2.$$

3. RELATIVE INTEGRAL BASES FOR $O_6(\sqrt[6]{-3})/O_3(\sqrt[3]{-3})$.

Let $O_6 = (1, \alpha)O_3$ for $\alpha \in O_6$. Again by theorem [6]

$$\text{disc}(1, \alpha) = \begin{vmatrix} 1 & \alpha \\ 1 & -\alpha \end{vmatrix} = 4\alpha^2 = d_{K_6/k_3} = \sqrt[3]{-3},$$

Note $\alpha = \frac{\sqrt[6]{-3}}{2} \notin O_6$, because $N_{6/3}(\alpha) = \frac{\sqrt[6]{-3}}{2} \cdot \frac{-\sqrt[6]{-3}}{2} = \frac{-\sqrt[3]{-3}}{4} \in O_3$. Hence, $(1, \alpha)$ is not a relative integral bases.

We define $\alpha = \frac{\beta + \sqrt[3]{-3}}{2}$ for $\beta \in O_3$ such that $N_{6/3}(\alpha)$ is divisible by $2 \cdot 2 = 4$ and $\alpha \in O_6$. If we take $\beta = \sqrt[3]{(-3)^2} \in O_3$, it satisfies the conditions, this is because

$$\frac{\beta + \sqrt[6]{-3}}{2} \cdot \frac{\beta - \sqrt[6]{-3}}{2} = \frac{\sqrt[3]{(-3)^4} - \sqrt[6]{(-3)^2}}{4} = \sqrt[3]{-3} \in O_3, \text{ by theorem [6],}$$

Also, $\text{disc}(1, \alpha) = d_{K_6/k_3}$, so that:

$$O_6 = \left(1, \frac{\sqrt[3]{(-3)^2} + \sqrt[6]{-3}}{2} \right) \cdot O_3.$$

4. RELATIVE INTEGRAL BASES FOR $O_6(\sqrt[6]{-3})/Z$.

Since $K_6 = Q(\sqrt[6]{-3})$, at first we start by:

$$O_6 = (1, \theta, \theta^2, \theta^3, \theta^4, \theta^5) Z$$

Let $\theta = \sqrt[6]{-3} \in O_6$. Since $\text{disc}(1, \theta, \theta^2, \theta^3, \theta^4, \theta^5) = 2^2 \cdot 2^2 \cdot 2^2 \cdot d_{K_6/k_1}$, we can apply

theorem [3] in order to cancel out $2^2 \cdot 2^2 \cdot 2^2$ and generate a new bases.

We will build a new bases $\alpha_1^* = \{\alpha_i: 0 \leq i \leq 5\}$. By the theorem [3] we check which α_i is going to be changed. $\alpha_0^* = \alpha_0/2 = 1/2 \notin 0_6$. Thus there is no change for the first bases element $\alpha_0 = 1$.

$$\alpha_1^* = \frac{g_1 \alpha_0 + \alpha_1}{2} = \frac{g_1 \alpha_0 + \theta}{2} \quad \text{for } 0 \leq g_1 \leq 1. \text{ For any value of } g_1, \alpha_1^* \text{ is not in } 0_6.$$

This is because

$$N_{6/3}(\alpha_1^*) = \frac{1+6\sqrt{-3}}{2} \cdot \frac{1-6\sqrt{-3}}{2} = \frac{1-3\sqrt{-3}}{4} \notin 0_3 \text{ and also since } N_{6/3}(\theta/2) \notin 0_3, \text{ so there is no change for } \alpha_1.$$

$$\alpha_2^* = \frac{g_1 \alpha_0 + g_2 \alpha_1 + \alpha_2}{2} \quad \text{for } 0 \leq g_1 \leq 1. \text{ For any value of } g_1, \alpha_2^* \notin 0_6, \text{ then there will be no change for } \alpha_2.$$

$$\alpha_3^* = \frac{g_1 \alpha_0 + g_2 \alpha_1 + g_3 \alpha_2 + \alpha_3}{2} \quad \text{for } 0 \leq g_i \leq 1. \text{ In this case for } g_1 = g_2 = g_3 = 1,$$

$$\alpha_3^* = 6\sqrt{(-3)^4} \in 0_6. \text{ This is because:}$$

$$\alpha_3^* = \frac{1+6\sqrt{(-3)^3}}{2} \cdot \frac{1-6\sqrt{(-3)^3}}{2} = \frac{1-6\sqrt{(-3)^6}}{4} = 1 \in 0_3, \text{ and for other values}$$

of $g_i, \alpha_3^* \notin 0_6$.

$$\alpha_4^* = \frac{g_1 \alpha_0 + g_2 \alpha_1 + g_3 \alpha_2 + g_4 \alpha_3^* + \alpha_4}{2}. \text{ In this case for } g_2 = g_4 = 1,$$

$$\alpha_4^* = \frac{6\sqrt{-3} + 6\sqrt{(-3)^4}}{2} \in 0_6. \text{ This is because}$$

$$N_{6/3}(\alpha_4^*) = \frac{6\sqrt{-3} + 6\sqrt{(-3)^4}}{2} \cdot \frac{6\sqrt{-3} - 6\sqrt{(-3)^4}}{2} = \frac{4 \cdot 3\sqrt{-3}}{4} \in 0_3, \text{ and for other } g_i, \alpha_4^* \notin 0_6.$$

$$\alpha_5^* = \frac{g_1 \alpha_0 + g_2 \alpha_1 + g_3 \alpha_2 + g_4 \alpha_3^* + g_5 \alpha_4^* + \alpha_5}{2}, \text{ for } g_2 = g_5 = 1,$$

$$\alpha_5^* = \frac{6\sqrt{(-3)^2} + 6\sqrt{(-3)^5}}{2} \in 0_6. \text{ This is because } N_{6/3}(\alpha_5^*) \in 0_3, \text{ and for other values}$$

of $g_i, \alpha_5^* \notin 0_6$. This last assertion is since

$$\text{disc}(\alpha_0, \alpha_1, \alpha_2, \alpha_3^*, \alpha_4^*, \alpha_5^*) = \frac{2^2 \cdot 2^2 \cdot 2^2}{2^2 \cdot 2^2 \cdot 2^2} \cdot d_{K6/k1}, \text{ and each } \alpha_i, \alpha_i^* \text{ are in } 0_6, \text{ then}$$

by theorem [6].

$$0_6 = \left(1, 6\sqrt{-3}, 6\sqrt{(-3)^2}, \frac{1+6\sqrt{(-3)^3}}{2}, \frac{6\sqrt{-3} + 6\sqrt{(-3)^4}}{2}, \frac{6\sqrt{(-3)^2} + 6\sqrt{(-3)^5}}{2} \right) \cdot \mathbb{Z}.$$

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