CONVERGENCE OF THE SOLUTIONS FOR THE EQUATION $x^{(iv)} + ax'' + bx' + g(x) + h(x) = p(t, x, \dot{x}, \ddot{x})$

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ABSTRACT. This paper is concerned with differential equations of the form

$$x^{(iv)} + ax'' + bx' + g(x) + h(x) = p(t, x, \dot{x}, \ddot{x})$$

where a, b are positive constants and the functions g, h and p are continuous in their respective arguments, with the function h not necessarily differentiable. By introducing a Lyapunov function, as well as restricting the incrementary ratio $\eta^{-1}\{h(\xi + \eta) - h(\xi)\}$, ($\eta \neq 0$), of h to a closed sub-interval of the Routh-Hurwitz interval, we prove the convergence of solutions for this equation. This generalizes earlier results.

KEY WORDS AND PHRASES. Routh-Hurwitz interval, Lyapunov function.

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1. INTRODUCTION.

Consider fourth-order differential equations of the form:

$$x^{(iv)} + ax'' + bx' + g(x) + h(x) = p(t, x, \dot{x}, \ddot{x}) \quad (1.1)$$

in which $a > 0, b > 0$, functions g and h are continuous in their respective arguments. The function $p(t, x, \dot{x}, \ddot{x})$ is assumed to have the form $q(t) + r(t, x, \dot{x}, \ddot{x})$ with the functions q and r depending explicitly on the arguments displayed, and continuous in their respective arguments. Further, we shall assume that $r(t, 0, 0, 0) = 0$ for all t .

The solutions of (1.1) will be said to converge if any two solutions $x_1(t), x_2(t)$ of (1.1) satisfy

$$\begin{aligned} x_2(t) - x_1(t) &\rightarrow 0, \quad \dot{x}_2(t) - \dot{x}_1(t) \rightarrow 0 \\ \ddot{x}_2(t) - \ddot{x}_1(t) &\rightarrow 0, \quad \ddot{\ddot{x}}_2(t) - \ddot{\ddot{x}}_1(t) \rightarrow 0, \end{aligned} \quad (1.2)$$

as $t \rightarrow \infty$.

The convergence of solutions for equations of the form (1.1) was earlier shown in [1], when $g(x) = cx$, with $c > 0$, and with the assumption that $h(x)$ is not necessarily differentiable, but with an incrementary ratio $\eta^{-1}\{h(\xi + \eta) - h(\xi)\}$,

($\eta \neq 0$), lying in a closed sub-interval I_0 of the Routh-Hurwitz interval $(0, (ab - c)/a^2)$, where

$$I_0 \equiv \left[\Delta_0, \frac{K(ab - c)c}{a^2} \right] \quad (1.3)$$

$\Delta_0 > 0$ and $K < 1$.

The main purpose of the present investigation is to give fourth-order analogues of [2], as well as extending earlier results in [1] to equations of the form (1.1) with the additional condition that for $y_1 \neq y_2$,

$$c_0 \geq \frac{g(y_2) - g(y_1)}{y_2 - y_1} \geq c \quad (1.4)$$

for some constants $c_0 > 0$ and $c > 0$, satisfying

$$abc > c_0^2. \quad (1.5)$$

Moreover, while proving the convergence results for (1.1), we shall give a general estimate for the constant $K < 1$, from which a particular case is derived.

2. MAIN RESULTS.

The main results of this paper, which are in some respects fourth-order analogues of [2] and generalizations of [1], are the following:

THEOREM 1. Suppose that $g(0) = h(0)$ and that

- (i) there are constants $c > 0$, $c_0 > 0$ such that $g(y)$ satisfies inequalities (1.4) and (1.5);
- (ii) there are constants $\Delta_0 > 0$, $K < 1$ such that for any ξ, η , ($\eta \neq 0$), the incrementary ratio for h satisfies

$$\eta^{-1}\{h(\xi + \eta) - h(\xi)\} \text{ lies in } I_0 \quad (2.1)$$

with I_0 as defined in (1.3);

- (iii) there is a continuous function $\phi(t)$ such that

$$\begin{aligned} & |r(t, x_2, y_2, z_2, w_2) - r(t, x_1, y_1, z_1, w_1)| \\ & \leq \phi(t)\{|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1| + |w_2 - w_1|\} \end{aligned} \quad (2.2)$$

holds for arbitrary $t, x_1, y_1, z_1, w_1, x_2, y_2, z_2$, and w_2 .

Then, there exists a constant D_1 such that if

$$\int_0^t \phi^\alpha(\tau) d\tau \leq D_1 t \quad (2.3)$$

for some α , in the range $1 < \alpha < 2$, then all solutions of (1.1) converge.

A very important step in the proof of Theorem 1 will be to give estimates for any two solutions of (1.1). This in itself, being of independent interest, is given as:

THEOREM 2. Let $x_1(t)$, $x_2(t)$ be any two solutions of (1.1). Suppose that all the conditions of Theorem 1 are satisfied, then for each fixed α , in the range $1 < \alpha < 2$, there exist constants D_2 , D_3 and D_4 such that for $t_2 > t_1$,

$$S(t_2) \leq D_2 S(t_1) \exp[-D_3(t_2 - t_1) + D_4 \int_{t_1}^{t_2} \phi^\alpha(\tau) d\tau] \quad (2.4)$$

where

$$S(t) = \{ [x_2(t) - x_1(t)]^2 + [\dot{x}_2(t) - \dot{x}_1(t)]^2 + [\ddot{x}_2(t) - \ddot{x}_1(t)]^2 + [\ddot{\ddot{x}}_2(t) - \ddot{\ddot{x}}_1(t)]^2 \}. \quad (2.5)$$

If we put $x_1(t) = 0$ and $t_1 = 0$, we immediately obtain:

COROLLARY 1. If $p = 0$ and the hypotheses (i) and (ii) of Theorem 1 hold, then the trivial solution of (1.1) is exponentially stable in the large.

Further, if we put $\xi = 0$ in (2.1) with η ($\eta \neq 0$) arbitrary, we obtain:

COROLLARY 2. If $p = 0$ and the hypotheses (i) and (ii) hold for arbitrary η ($\eta \neq 0$), and $\xi = 0$, then there exists a constant $D_5 > 0$ such that every solution $x(t)$ of (1.1) satisfies

$$|x(t)| \leq D_5; |\dot{x}(t)| \leq D_5; |\ddot{x}(t)| \leq D_5; |\ddot{\ddot{x}}(t)| \leq D_5. \quad (2.6)$$

3. PRELIMINARY RESULTS.

Let $Q(t) = \int_0^t q(\tau) d\tau$. For convenience, by setting $\dot{x} = y$, $\dot{y} = z$ and $\dot{z} = w + Q(t)$, we replace equation (1.1) by the equivalent system:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= w + Q(t) \\ \dot{w} &= -aw - bz - g(y) - h(x) + r(t, x, y, z, w + Q(t)) - aQ(t) \end{aligned} \quad (3.1)$$

Let $(x_i(t), y_i(t), z_i(t), w_i(t))$, ($i = 1, 2$), be two solutions of (3.1), such that

$$c \leq \frac{g(y_2) - g(y_1)}{y_2 - y_1} \leq c_0; \quad (3.2)$$

and

$$\Delta_0 \leq \frac{h(x_2) - h(x_1)}{x_2 - x_1} \leq \frac{K(ab - c)c}{a^2} \quad (3.3)$$

where c , c_0 , Δ_0 , K are as defined in (1.3), (1.4) and (1.5).

Our main tool in the proofs of the convergence Theorems will be the following function: $W = W(x_2 - x_1, y_2 - y_1, z_2 - z_1, w_2 - w_1)$ defined by

$$\begin{aligned} 2W &= \{c^2 \varepsilon (1 - \varepsilon) (x_2 - x_1)^2 + ac(1 - \varepsilon) (D - 1) (y_2 - y_1)^2 + \\ &\quad + 2c[\varepsilon + (D - 1)] (y_2 - y_1) (z_2 - z_1) + \varepsilon D (w_2 - w_1)^2 + \\ &\quad + b(D - 1) (z_2 - z_1)^2 + [(1 - \varepsilon)D - 1][a(z_2 - z_1) + (w_2 - w_1)]^2 + \\ &\quad + [c(1 - \varepsilon) (x_2 - x_1) + b(y_2 - y_1) + (z_2 - z_1) + (w_2 - w_1)]^2\}, \end{aligned} \quad (3.4)$$

where $D - 1 = (\delta + c\varepsilon)/(ab - c - \delta)$, with $ab - c > \delta > 0$; $0 < \varepsilon < 1$; and $ab\varepsilon(2 - \varepsilon) = \delta$. This is an adaptation of the function V used in [1].

Since $0 < \varepsilon < 1$, following the argument used in [1], we can easily verify the following for W .

LEMMA 1. (i) $W(0,0,0,0) = 0$; and

(ii) there exist finite constants $D_6 > 0$, $D_7 > 0$ such that

$$\begin{aligned} D_6 \{ (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (w_2 - w_1)^2 \} &\leq W \leq \\ &\leq D_7 \{ (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (w_2 - w_1)^2 \}. \end{aligned} \quad (3.5)$$

If we define the function $W(t)$ by $W(x_2(t) - x_1(t), y_2(t) - y_1(t), z_2(t) - z_1(t), w_2(t) - w_1(t))$, and using the fact that the solutions $(x_i, y_i, z_i, w_i + Q(t))$, $(i = 1, 2)$, satisfy (3.1), then $S(t)$ as defined in (2.5) becomes

$$\begin{aligned} S(t) = \{ [x_2(t) - x_1(t)]^2 + [y_2(t) - y_1(t)]^2 + [z_2(t) - z_1(t)]^2 + \\ + [w_2(t) - w_1(t)]^2 \} \end{aligned} \quad (3.6)$$

We can then prove the following result on the derivative of $W(t)$ with respect to t .

LEMMA 2. Let the hypotheses (i) and (ii) of Theorem 1 hold. Then, there exist positive finite constants D_8 and D_9 such that

$$\frac{dW}{dt} \leq -2D_8 S + D_9 S^{\frac{1}{2}} |\Theta| \quad (3.7)$$

where $\Theta = r(t, x_2, y_2, z_2, w_2 + Q) - r(t, x_1, y_1, z_1, w_1 + Q)$.

PROOF OF LEMMA 2. On using (3.1), a direct computation of $\frac{dW}{dt}$ gives after simplification

$$\frac{dW}{dt} = -W_1 + W_2 \quad (3.8)$$

where

$$\begin{aligned} W_1 = \{ c(1-\varepsilon)H(x_2, x_1)(x_2 - x_1)^2 + bc\varepsilon(y_2 - y_1)^2 + ab\varepsilon(1-\varepsilon)D(z_2 - z_1)^2 \\ + a\varepsilon D(w_2 - w_1)^2 \} + \{ G(y_2, y_1) - c \} \{ c(1-\varepsilon)(x_2 - x_1) + \\ + b(y_2 - y_1) + a(1-\varepsilon)D(z_2 - z_1) + D(w_2 - w_1) \} (y_2 - y_1) + \\ + H(x_2, x_1) \{ b(y_2 - y_1) + a(1-\varepsilon)D(z_2 - z_1) + D(w_2 - w_1) \} (x_2 - x_1) \end{aligned}$$

and

$$W_2 = \Theta(t) \{ c(1-\varepsilon)(x_2 - x_1) + b(y_2 - y_1) + a(1-\varepsilon)D(z_2 - z_1) + D(w_2 - w_1) \},$$

with

$$G(y_2, y_1) = \frac{g(y_2) - g(y_1)}{y_2 - y_1}, \quad (y_2 \neq y_1); \quad (3.9)$$

$$H(x_2, x_1) = \frac{h(x_2) - h(x_1)}{x_2 - x_1}, \quad (x_2 \neq x_1). \quad (3.10)$$

Let $\lambda = \{G(y_2, y_1) - c\} > 0$ for $y_2 \neq y_1$. Define

$$\sum_{i=1}^5 \alpha_i = 1; \quad \sum_{i=1}^5 \beta_i = 1; \quad \sum_{j=1}^3 \gamma_j = 1 \quad \text{and} \quad \sum_{j=1}^3 \delta_j = 1,$$

with $\alpha_i > 0$, $\beta_i > 0$, $\gamma_j > 0$ and $\delta_j > 0$. Further, let us denote $H(x_2, x_1)$ simply by H . Then, we can re-arrange W_1 as

$$W_1 = W_{11} + W_{12} + W_{13} + W_{14} + W_{21} + W_{23} + W_{24} \quad (3.11)$$

where

$$W_{11} = \{\alpha_1 c(1-\epsilon)H(x_2-x_1)^2 + b(\beta_1 c\epsilon + \lambda)(y_2-y_1)^2 + \gamma_1 ab\epsilon(1-\epsilon)D(z_2-z_1)^2 + \delta_1 a\epsilon D(w_2-w_1)^2\}$$

$$W_{12} = \{\beta_2 bc\epsilon(y_2-y_1)^2 + \lambda c(1-\epsilon)(x_2-x_1)(y_2-y_1) + \alpha_2 c(1-\epsilon)H(x_2-x_1)^2\};$$

$$W_{23} = \{\beta_3 bc\epsilon(y_2-y_1)^2 + \lambda a(1-\epsilon)D(y_2-y_1)(z_2-z_1) + \gamma_2 ab\epsilon(1-\epsilon)D(z_2-z_1)^2\};$$

$$W_{24} = \{\beta_4 bc\epsilon(y_2-y_1)^2 + \lambda D(y_2-y_1)(w_2-w_1) + \delta_2 a\epsilon D(w_2-w_1)^2\};$$

$$W_{12} = \{\alpha_3 c(1-\epsilon)H(x_2-x_1)^2 + bH(x_2-x_1)(y_2-y_1) + \beta_5 bc\epsilon(y_2-y_1)^2\};$$

$$W_{13} = \{\alpha_4 c(1-\epsilon)H(x_2-x_1)^2 + a(1-\epsilon)DH(x_2-x_1)(z_2-z_1) + \beta_3 ab\epsilon(1-\epsilon)D(z_2-z_1)^2\};$$

$$\text{and} \quad W_{14} = \{\alpha_5 c(1-\epsilon)H(x_2-x_1)^2 + DH(x_2-x_1)(w_2-w_1) +$$

Each W_{ij} , ($i \neq j$), ($i = 1, 2$; $j = 1, 2, 3, 4$), is quadratic in its respective variables. Also, using the fact that any quadratic of the form $Au^2 + Buv + Cv^2$ is non-negative if $(4AC - B^2) > 0$, we obtain that

$$W_{21} \geq 0 \quad \text{if} \quad \lambda^2 \leq \frac{4b\epsilon\Delta\alpha_2\beta_2}{1-\epsilon};$$

$$W_{23} \geq 0 \quad \text{if} \quad \lambda^2 \leq \frac{4b^2c\epsilon^2\beta_3\gamma_2}{a(1-\epsilon)D};$$

$$W_{24} \geq 0 \quad \text{if} \quad \lambda^2 \leq \frac{4abc\epsilon^2\delta_2\beta_4}{D};$$

$$W_{12} \geq 0 \quad \text{if} \quad H \leq \frac{4c^2\epsilon(1-\epsilon)\alpha_3\beta_5}{b};$$

$$W_{13} \geq 0 \quad \text{if} \quad H \leq \frac{4bc\epsilon\alpha_4\gamma_3}{aD};$$

$$\text{and} \quad W_{14} \geq 0 \quad \text{if} \quad H \leq \frac{4ac\epsilon(1-\epsilon)\alpha_5\delta_3}{D}.$$

Thus $W_1 > W_{11}$, provided that

$$0 \leq \lambda^2 \leq 4 \min \left\{ \frac{b\epsilon\Delta_0\alpha_2\beta_2}{(1-\epsilon)} ; \frac{b^2c\epsilon\beta_3\gamma_2}{a(1-\epsilon)D} ; \frac{abc\epsilon^2\delta_2\beta_4}{D} \right\} \quad (3.12)$$

and

$$H \text{ lies in } I_0 \equiv \left[\Delta_0, \frac{K(ab-c)c}{a^2} \right] \quad (3.13)$$

a closed sub-interval of the Routh-Hurwitz interval $(0, (ab-c)c/a^2)$, with

$$K = \left(\frac{4}{ab-c} \right) \min \left\{ \frac{ca^2\epsilon(1-\epsilon)\alpha_3\beta_5}{b} ; \frac{ab\epsilon\alpha_4\gamma_3}{D} ; \frac{a^3\epsilon(1-\epsilon)\alpha_5\delta_3}{D} \right\} \quad (3.14)$$

< 1 .

By choosing $2D_8 = \min\{c(1-\epsilon)\Delta_0; bc\epsilon; ab\epsilon(1-\epsilon)D; a\epsilon D\}$, we clearly have

$$W_1 \geq W_{11} \geq 2D_8S. \quad (3.15)$$

also, if we choose $D_9 = 2 \max\{c(1-\epsilon); b; a(1-\epsilon)D; D\}$, we obtain:

$$W_2 \leq D_9S^{\frac{1}{2}}|\theta|. \quad (3.16)$$

Combining (3.15) and (3.16) in (3.8), we obtain (3.7). This completes the proof of Lemma 2.

4. PROOF OF THEOREM 2.

This follows directly from [3], on using inequality (3.7). Let α be any constant in the range $1 \leq \alpha \leq 2$. Set $2\mu = 2 - \alpha$, so that $0 \leq 2\mu \leq 1$. We re-write (3.7) in the form

$$\frac{dW}{dt} + D_8S \leq D_9S^{\frac{1}{2}}W^*$$

$$\text{where } W^* = (|\theta| - D_8D_9^{-1}S^{\frac{1}{2}})S^{\frac{1}{2}-\mu}. \quad (4.1)$$

Considering the two cases (i) $|\theta| < D_8S^{1/2}D_9$ and (ii) $|\theta| > D_8S^{1/2}D_9$ separately, we find that in either case, there exists some constant $D_{11} > 0$ such that $W^* < D_{11}|\theta|^{2(1-\mu)}$. Thus using (2.2), inequality (4.1) becomes

$$\frac{dW}{dt} + D_8S \leq D_{12}S^{\frac{1}{2}\phi}S^{2(1-\mu)}S^{(1-\mu)}, \quad (4.2)$$

where $D_{12} > 2D_9D_{11}$. This immediately gives

$$\frac{dW}{dt} + (D_{13} - D_{14}\phi^\alpha(t))W \leq 0 \quad (4.3)$$

after using Lemma 1 on W , with D_{13} and D_{14} as some positive constants.

On integrating (4.3) from t_1 to t_2 , ($t_2 > t_1$), we obtain

$$W(t_2) \leq W(t_1) \exp \{-D_{13}(t_2-t_1) + D_{14} \int_{t_1}^{t_2} \phi^\alpha(\tau) d\tau\}. \quad (4.4)$$

Again, using Lemma 1, we obtain (2.4), with $D_2 = D_7/D_6$, $D_3 = D_{13}$ and $D_4 = D_{14}$. This completes the proof of Theorem 2.

5. PROOF OF THEOREM 1.

This follows from the estimate (2.4) and the condition (2.3) on $\phi(t)$. Choose $D_1 = D_3/D_4$ in (2.3). Then, as $t = (t_2 - t_1) \rightarrow \infty$, $S(t) \rightarrow 0$, which proves that as $t \rightarrow \infty$,

$$x_2(t) - x_1(t) \rightarrow 0, \quad \dot{x}_2(t) - \dot{x}_1(t) \rightarrow 0,$$

$$\ddot{x}_2(t) - \ddot{x}_1(t) \rightarrow 0, \quad \ddot{\ddot{x}}_2(t) - \ddot{\ddot{x}}_1(t) \rightarrow 0.$$

This completes the proof of Theorem 1.

6. REMARKS.

(i) If in (3.14) we choose

$$\alpha_1 = 1/2; \quad \alpha_j = 1/8 \quad (j = 2, 3, 4, 5);$$

$$\beta_1 = 1/2; \quad \beta_j = 1/8 \quad (j = 2, 3, 4, 5);$$

$$\gamma_1 = 1/2; \quad \gamma_2 = \gamma_3 = 1/4;$$

$$\delta_1 = 1/2; \quad \delta_2 = \delta_3 = 1/4,$$

we obtain

$$K = \left(\frac{1}{16(ab-c)} \right) \min \left\{ \frac{ca^2\varepsilon(1-\varepsilon)}{b}; \frac{2ab\varepsilon}{D}; \frac{2a^3\varepsilon(1-\varepsilon)}{D} \right\} < 1.$$

(ii) As remarked in [1], the results remain valid if we replace $\phi(t)$ in (2.3) by a constant $D_{15} > 0$.

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