ON CONVEX FUNCTIONS OF ORDER \( \alpha \) AND TYPE \( \beta \)

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ABSTRACT. Owa \([1]\) gave three subordination theorems for convex functions of order \( \alpha \) and starlike functions of order \( \alpha \). Unfortunately, none of the theorems is correct. In this paper, similar problems are discussed for a generalized class and sharp results are given.

KEY WORDS AND PHRASES. Subordination, Hadamard product, convex functions of order \( \alpha \) and type \( \beta \).

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1. INTRODUCTION.

Let \( f(z) \) and \( g(z) \) be analytic in the unit disk \( D = \{ z : |z| < 1 \} \). \( f(z) \) is said to be subordinate to \( g(z) \), denoted by \( f(z) \prec g(z) \), if there exists a function \( w(\tau) \) analytic and satisfying \( |w(z)| \leq |z| \) in \( D \) such that \( f(z) = g(w(z)) \). If \( f(z) \prec g(z) \) is equivalent to \( f(0) = g(0) \) and \( f(D) \subseteq g(D) \), then \( f(z) \prec g(z) \) is valent in \( D \), then \( f(z) - g(z) \) is equivalent to \( f(0) = g(0) \) and \( f(D) \subseteq g(D) \).

Let \( S*(\alpha, \beta) \) be the family of starlike functions of order \( \alpha \) and type \( \beta \). That is, it consists of analytic functions \( f(z) = z + a_2 z^2 + \ldots \) satisfying

\[
|zf'(z)/f(z)-1| < \left| (2\beta - 1)zf'(z)/f(z) + 1 - 2\beta \right| \quad (z \in D),
\]

where \( 0 \leq \alpha < 1 \) and \( 0 < \beta \leq 1 \). This class was first introduced by Juneja and Mogra \([2]\). It is clear that \( S*(\alpha, 1) = S*(\alpha) \), the usual class of starlike functions of order \( \alpha \).

Similarly, we define the following general class.

DEFINITION. An analytic function \( f(z) = z + a_2 z^2 + \ldots \) is called convex of order \( \alpha \) and type \( \beta \) if and only if

\[
|zf''(z)/f'(z)| < \left| (2\beta - 1)zf''(z)/f'(z) + 2\beta(1 - \alpha) \right| \quad (z \in D).
\]

The class of these functions is denoted by \( K(\alpha, \beta) \).

\( K(\alpha, 1) \) is the well known class \( K(\alpha) \), which consists of convex functions of order \( \alpha \). It is easily seen that \( f(z) \in K(\alpha, \beta) \) if and only if \( zf'(z) \in S*(\alpha, \beta) \).

We got the following theorem in \([3]\).

THEOREM A. Let \( f(z) \in S*(\alpha, \beta) \), then we have

\[
zf'(z)/f(z) \prec \frac{(1 + (1 - 2\beta)z)/(1 + (1 - 2\beta)z)}{1},
\]

(1.3)
All of the results are best possible.

In section 2 of this paper, we give a counterexample of Owa's results and point out the mistakes in [1]. Then we discuss similar problems for the class $K(a, \beta)$ and get sharp subordination and convolution theorems. And we give a characterization for convex functions of order $a$ and type $\beta$ in section 3. Finally, we obtain some inequalities by using the subordination results.

2. A COUNTEREXAMPLE.

Theorem 1 in [1] is equivalent to that if $f(z) \in K(a)$, then

$$f'(z) \prec e^{-4(1-a)/(1-z)} = F(z), \quad (2.1)$$

and if $f'(r e^{i \theta})$ lies for some $r \neq 0$ on the boundary of $F(|z| < r)$ if and only if

$$f(z) = \int_0^z e^{4(1-a)/(1-\varepsilon t)} dt \quad (|\varepsilon| = 1). \quad (2.2)$$

It is well known that

$$f(z) = \int_0^z (1-t)^{-2(1-a)} dt \in K(a).$$

(2.1) implies that

$$(1-z)^{-2(1-a)} \prec e^{-4(1-a)/(1-z)},$$

or equivalently,

$$\log(1-z) \prec 2/(1-z) \quad (2.3)$$

where log is to be the branch which vanishes at the point one. But clearly, (2.3) does not hold.

The mistake arises from that

$$zf''(z)/f'(z) \prec 4(1-a)z/(1-z)^2 \quad (2.4)$$

implies

$$\log f'(z) \prec -4(1-a)/(1-z).$$

In fact, from (2.4) we can only get

$$\log f'(z) \prec 4(1-a)z/(1-z).$$

And (2.2) was got from $f(0) = f'(0) - 1 = 0$ and

$$zf''(z)/f'(z) = 4(1-a)\varepsilon z/(1-\varepsilon z)^2.$$

So it should be

$$f(z) = \int_0^z \exp[4(1-a)\varepsilon t/(1-\varepsilon t)] dt.$$
Furthermore, this function does not belong to $K(\alpha)$.

There are similar mistakes in the theorem 2 [1]. And the family of functions which satisfy the conditions in theorem 3 [1] is empty since $\text{Re}(zf'(z)) = 0$ at $z = 0$. Therefore this theorem is meaningless. From the proof of the theorem 3, maybe the condition should be $\text{Re}(zf'(z)) < \alpha$, not $\text{Re}(zf'(z)) > \alpha$. If so, the following proof goes wrong again. There are also several places needed to be corrected. We omit them here.

3. SUBORDINATION AND CONVOLUTION THEOREMS.

The leading element of $K(\alpha, \beta)$ is

$$k(\alpha, \beta, z) = \begin{cases} 
(1 - 2\beta)^{-1} \{(1 + (1 - 2\beta)z)/(1 - 2\beta)\} & (\beta \neq \frac{1}{2}, \alpha \neq \frac{1}{2}/\beta) \\
\text{log}(1 + (1 - 2\beta)z) & (\beta \neq \frac{1}{2}, \alpha = \frac{1}{2}/\beta) \\
(e^{(1-\alpha)z}/(1 - \alpha) & (\beta = \frac{1}{2})
\end{cases}$$

(3.1)

It is not difficult to prove that for an analytic function $f(z) = z + a_2z^2 + ...$, (1.3) implies that $f(z) \in S^*(\alpha, \beta)$. From Theorem A and the correspondence between $K(\alpha, \beta)$ and $S^*(\alpha, \beta)$, we have the following

**THEOREM 1.** $f(z) \in K(\alpha, \beta)$ if and only if $f(z) = z + a_2z^2 + ...$ is analytic in $D$ and

$$zf''(z)/f'(z) \leq 2\beta(1 - \alpha)z/(1 + (1 - 2\beta)z).$$

(3.2)

Moreover, let $f(z) \in K(\alpha, \beta)$, then we have sharp subordinations

$$f'(z) \preceq (1 + (1 - 2\beta)z)^{2\beta(1-\alpha)/(1-2\beta)} (\beta \neq \frac{1}{2}),$$

$$f'(z) \preceq e^{(1-\alpha)z} (\beta = \frac{1}{2}).$$

The first result of Theorem 1 is equivalent to that an analytic function $f(z) = z + a_2z^2 + ... \in K(\alpha, \beta)$ if and only if

$$1 + zf''(z)/f'(z) \in Q(\alpha, \beta) (z \in D),$$

(3.3)

where

$$Q(\alpha, \beta) = \begin{cases} 
\{w; |w - \alpha - (1-\alpha)/(1-\beta)| < \frac{1}{2}(1-\alpha)/(1-\beta)\} (\beta < 1) \\
\{w; \text{Re} w > \alpha\} (\beta = 1).
\end{cases}$$

**COROLLARY 1.** $K(\alpha, \beta_1) \subseteq K(\alpha, \beta_2) \subseteq K(\alpha, 1) = K(\alpha)$ if $\beta_1 \leq \beta_2 \leq 1$.

$K(\alpha_1, \beta) \subseteq K(\alpha_2, \beta) \subseteq K(0, \beta)$ if $\alpha_1 \geq \alpha_2 \geq 0$.

**THEOREM 2.** Let $p(z) \in K = K(0, 1)$, and $f(z) \in K(\alpha, \beta)$, then

$$p*f(z) \in K(\alpha, \beta),$$

where $*$ denotes the Hadamard product.

**PROOF.** We know that $1 + zf''(z)/f'(z) = (zf'(z))'/f'(z)$ takes all its values in the convex domain $Q(\alpha, \beta)$. A result of Ruscheweyh and Sheil-Small [4] implies that
p(z)*(zf'(z))'/p(z)*(zf'(z)) also takes all its values in Q(0,1) since we have f(z) ∈ K(0,1) = K from Corollary 1. It is easy to see that

\[ p(z)*(zf'(z)) = z(p*f)'(z), \]
\[ p(z)*\{z(zf'(z))'\} = z\{p(z)*(zf')(z)\}' = z(p*f)'(z) + z^2(p*f)''(z). \]

Thus for each z ∈ D

\[ l+z(p*f)''(z)/(p*f)'(z) = p(z)*(zf'(z))'/p(z)*(zf'(z)) \in Q(0,1), \]

which yields p*f(z) ∈ K(0,1). The proof is completed.

For α = 0 and β = 1, Theorem 2 is the well known Polya-Schoenberg conjecture proved in [4].

**COROLLARY 2.** K(0,1) ⊂ S*(0,1).

**PROOF.** If f(z) ∈ K(0,1). Let p(z) = log(1-z)^{-1} in Theorem 2, we get

\[ g(z) = \int_0^z f(t)/t \, dt \in K(0,1), \]

which gives that f(z) = zg'(z) ∈ S*(0,1).

**LEMMA 1.** g(z) = k(α,β,z)/zk'(α,β,z) is an analytic and convex univalent function in D. Moreover, g(z) is analytic and univalent on D except for z=1 when β=1 for which

\[ \lim_{z \to 1} G(z) = \infty. \]

**PROOF.** We may assume that β ≠ 1 and α ≠ 1/β since the convexity for β ≠ 1 or α ≠ 1/β can be deduced from the convexity for β ≠ 1/β, α ≠ 1/β.

From (3.1), we find that

\[ G(z) = (2β-1+G_1(z))/(2β-1), \]

where

\[ G_1(z) = z^{-1}(1+(1-2β)z)2β(1-α)/(2β-1) - 1. \]

So we have

\[ G_1(z) + 2β(1-α) = 2β(1-α)/z \int_0^z G_2(t) \, dt, \]

where

\[ G_2(z) = 1-(1+(1-2β)z)(1-2βa)/(2β-1). \]

Hence

\[ 1+zG_2''(z)/G_2'(z) = (1-(1-2βa)z)/(1+(1-2β)z), \]

which yields that G_2(z) is a convex univalent function. G_1(z) + 2β(1-α) is also convex follows from a result due to Libera [5]. Thus G_1(z) is convex, and so is G(z). This results in the conclusions as desired. When β = 1, we can get the other result easily. We come to the end of our proof.
THEOREM 3. Let \( f(z) \in K(\alpha, \beta) \), we have sharp subordination
\[
\frac{zf'(z)}{f(z)} < \frac{zk'(\alpha, \beta, z)}{k(\alpha, \beta, z)}.
\]
(3.4)

To prove Theorem 3, we need the following lemma due to Miller and Mocanu [6].

LEMMA A. Let \( q(z) = a + q_1z + \ldots \) be regular and univalent on \( \overline{D} \) except for those points \( \zeta \in \partial D \) for which \( \lim_{z \to \zeta} q(z) = \infty \), and let \( p(z) = a + p_1z + \ldots \) be analytic in \( D \) with \( p(z) \not= a \). If there exists a point \( z_0 \in D \) such that \( p(z_0) = q(\zeta) \) and \( p(|z| < |z_0|) \subset q(D) \). Then
\[
z_0p'(z_0) = m_0q'(\zeta),
\]
where \( q^{-1}(p(z_0)) = \zeta = e^{i\theta} \) and \( m \geq 1 \).

PROOF OF THEOREM 3. Let \( g(z) = f(z)/zf'(z) \), \( G(z) = k(\alpha, \beta, z)/zk'(\alpha, \beta, z) \) and \( H(z) = i/G(z) \). The required result is equivalent to that
\[
g(z) \preceq G(z).
\]
(3.5)

It is clear that (3.5) is to be the case if \( g(z) \equiv 1 \). So we assume that \( g(z) \not\equiv 1 \). We can easily check that
\[
1 + zf''(z)/f'(z) = 1/g(z) - zg'(z)/g(z),
\]
\[
1/G(z) - zG'(z)/G(z) = (1 + (1 - 2\alpha)z)/(1 + (1 - 2\beta)z).
\]

If (3.5) is not true, then by using Lemma 1 and Lemma A, there exists \( z_0 \in D \) such that
\[
z_0g'(z_0) = m_0G'(\zeta), \quad g(z_0) = G(\zeta),
\]
where \( |\zeta| = 1 \) and \( m \geq 1 \). Thus we have
\[
1 + z_0f''(z_0)/f'(z_0) = 1/G(\zeta) - m_0G'(\zeta)/G(\zeta)
= m(1/G(\zeta) - G'(\zeta)/G(\zeta) - (m-1)/G(\zeta)
= m(1 + (1 - 2\alpha)\zeta)/(1 + (1 - 2\beta)\zeta) - (m-1)H(\zeta).
\]
(3.6)

From Corollary 2, we know that \( k(\alpha, \beta, z) \in S^*(\alpha, \beta) \), which gives that
\[
H(\zeta) \in Q(\alpha, \beta).
\]
(3.7)

For \( \beta = 1 \), (3.7) is equivalent to that \( \Re H(\zeta) \geq \alpha \). Thus (3.6) implies
\[
\Re(1 + z_0f''(z_0)/f'(z_0)) = m\Re((1 + (1 - 2\alpha)\zeta)/(1 - \zeta)) - (m-1)\Re H(\zeta)
\leq m\alpha - (m-1)\alpha = \alpha,
\]
which contradicts that \( f(z) \in K(\alpha, \beta) \).

For \( \beta = \frac{1}{2} \), (3.7) becomes \( |H(\zeta)| - 1 \leq 1 - \alpha \). And it follows from (3.6)
\[
|z_0f''(z_0)/f'(z_0)| = |m(1 + (1 - \alpha)\zeta) - (m-1)H(\zeta)|
= |m(1 - \alpha)\zeta - (m-1)(H(\zeta) - 1)| \geq m(1 - \alpha) - (m-1)(1 - \alpha) = 1 - \alpha,
which is impossible since \( f(z) \notin K(\alpha, \beta) \).

For \( \beta \neq \frac{1}{2}, 1 \), (3.7) is the same as

\[
|H(\xi) - (1-\alpha)/(1-\beta) - 2(1-\beta)| \leq (1-\alpha)/(2(1-\beta)).
\]

We get from (3.6) that

\[
|1+z_f''(z_0)/f'(z_0)-\frac{1}{2}(1-\alpha)/(1-\beta) - (1-\alpha)/(1-\beta)|
\]

\[
= |m(1+(1-2\alpha)\zeta)/(1+(1-2\beta)\zeta)-\frac{1}{2}(1-\alpha)/(1-\beta) - (m-1)H(\xi)|
\]

\[
\geq \frac{m}{2}(1-(1-\alpha)/(1-\beta)) - (m-1)\frac{1}{2}(1-\alpha)/(1-\beta) = \frac{1}{2}(1-\alpha)/(1-\beta),
\]

which is also impossible. This completes the proof of Theorem 3.

For \( \beta = 1 \), (3.4) was first verified by MacGregor [7]. Our proof is much simpler than that in [7].

**THEOREM 4.** Let \( f(z) = z+a_2z^2+... \) be analytic in \( D \). Then \( f(z) \notin K(\alpha, \beta) \) if and only if

\[
\frac{1}{z} \left\{ f * z+a_2z^2+... (1-\alpha)/(1-\beta) \right\} \neq 0 \quad (|z| < 1, |x| = 1).
\]

**PROOF.** We only prove the result for \( \beta < 1 \). The result for \( \beta = 1 \) can be deduced from that for \( \beta < 1 \) by letting \( \beta \) tend to 1.

We know \( f(z) \notin K(\alpha, \beta) \) if and only if \( 1+z_f''(z)/f'(z) \notin Q(\alpha, \beta) \) \((z \in D)\). Since

\[
1+z_f''(z)/f'(z) = 1 \quad \text{at} \quad z = 0, 1+z_f''(z)/f'(z) \notin Q(\alpha, \beta) \]

is equivalent to

\[
1+z_f''(z)/f'(z) \neq (1-\alpha)/(1-\beta) \quad (|y| = 1),
\]

which simplifies to

\[
z_f''(z)+f'(z) (1-\alpha)/(1-\beta) \neq 0 \quad (|y| = 1). \quad (3.8)
\]

As

\[
f'(z) = \frac{f(z)}{z} \ast \frac{1}{(1-z)^2},
\]

\[
z_f''(z) = \frac{f(z)}{z} \ast \frac{2z}{(1-z)^3}.
\]

We have

\[
z_f''(z)+f'(z) (1-\alpha)/(1-\beta)
\]

\[
= \frac{f(z)}{z} \ast (2z/(1-z)^3 + (1-z)^{-2}(1-\alpha)/(1-\beta))
\]

\[
= (1-\alpha)(1-\beta)^{-1} \frac{f(z)}{z} \ast \left\{ \frac{1}{(1-z)^3} (1+(4(1-\beta)/(1-\alpha)(1-2\beta))(1-z)) \right\}
\]
It is not difficult to verify that \( \frac{1}{2}(1-a)(1-2\beta-y)/(1-\beta) \neq 0 \) and 
\[ 2(1-\beta)/(1-2\beta-y) = 1 - \frac{1}{2}(1-x)/\beta \]
is a homotopic mapping from \( |y| = 1 \) to \( |x| = 1 \). Thus (3.8) is equivalent to

\[
\frac{f(z)}{z} \ast \frac{1+z(\beta+\alpha-1+x)/(1-\alpha\beta)}{(1-z)^3} \neq 0 \quad (|x| = 1),
\]

which is the same as the result desired. This completes the proof of Theorem 4.

For \( \beta = 1 \), Theorem 4 was first given in [8].

4. APPLICATIONS.

With the help of principle of subordination, we can get the following results from Theorem 1 and Theorem 3. Here we omit their proof.

**THEOREM 5.** Let \( f(z) \in K(a,\beta) \) and \( |z| = r < 1 \), then we obtain sharp estimates.

\[
2\beta(1-a)r/((1+|1-2\beta|r) \leq |zf''(z)/f'(z)| \leq 2\beta(1-a)r/((1-|1-2\beta|r),
\]

\[
|\arg(1+zf''(z)/f'(z))| \leq \arcsin\left(2\beta(1-a)r/((1+(1-2\beta)(1-2\alpha)r^2)\right),
\]

\[
(1+(2\beta-1)r)\frac{2\beta(1-a)/(1-\beta)}{\beta \neq \frac{1}{2}},
\]

\[
eq |f'(z)| \leq (1-(2\beta-1)r)2\beta(1-a)/(1-2\beta) \quad (\beta \neq \frac{1}{2}),
\]

\[
eq e^{-(1-a)r} \leq |f'(z)| \leq e^{(1-a)r} \quad (\beta = \frac{1}{2}),
\]

\[
|\arg f'(z)| \leq (1-2\beta)^{-1}2\beta(1-a)\arcsin(1-2\beta)r \quad (\beta \neq \frac{1}{2}),
\]

\[
|\arg f'(z)| \leq (1-a)r \quad (\beta = \frac{1}{2}),
\]

\[
\min\left|zk'(a,\beta,z)/k(a,\beta,z)\right| \leq \left|zf'(z)/f(z)\right| \leq rk'(a,\beta,r)/k(a,\beta,r),
\]

\[
\min\left|zk'(a,\beta,z)/k(a,\beta,z)\right| \leq \max\left|zk'(a,\beta,z)/k(a,\beta,z)\right|.
\]

Using a traditional method, we get from Theorem 5 the following

**COROLLARY 3.** Let \( f(z) \in K(a,\beta) \) and \( |z| = r < 1 \), then we have sharp inequality

\[
-k(a,\beta,-r) \leq |f(z)| \leq k(a,\beta,r).
\]

Theorem 3 also has an application of getting the sharp order of starlikeness for the functions in \( K(a,\beta) \).

**COROLLARY 4.** If \( f(z) \in K(a,\beta) \), \( |z| = r < 1 \). Then

\[
\text{Re}\{zf'(z)/f(z)\} \geq \min_{|z|=r} \text{Re}\{zk'(a,\beta,z)/k(a,\beta,z)\}.
\]

In particular, we have \( f(z) \in s^\alpha(s(a,\beta)) \), where

\[
s(a,\beta) = \inf_{|z|<1} \text{Re}\{zk'(a,\beta,z)/k(a,\beta,z)\} > a.
\]

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