MODIFIED WHYBURN SEMIGROUPS

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ABSTRACT. Let $f: X \times Y$ be a continuous semigroup homomorphism. Conditions are given which will ensure that the semigroup $X \cup Y$ is a topological semigroup, when the modified Whyburn topology is placed on $X \cup Y$.

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1. INTRODUCTION.

Let $(X, m_1)$ and $(Y, m_2)$ be semigroups and let $f: X \times Y$ be a semigroup homomorphism. An associative multiplication $m$ may be defined on the disjoint union of $X$ and $Y$ as follows: $m$ is $m_1$ on $X$, $m_2$ on $Y$ and $m_2(f(x), y)$ if $x \in X$ and $y \in Y$. If we assume that $X$ and $Y$ are Hausdorff semigroups and that $f$ is continuous, then $m$ is continuous in the disjoint union (or direct sum) topology. Let $(X \cup Y, m)$ denote this Hausdorff semigroup.

Let $Z$ denote the disjoint union of $X$ and $Y$ with Whyburn's unified topology [1]; i.e., $V$ is open in $Z$ iff $V \cap X$ and $V \cap Y$ are open in $X$ and $Y$, respectively, and for any compact $K$ in $V \cap Y$, $f^{-1}(K) \cap V$ is compact. If $X$ is locally compact, then $Z$ is Hausdorff, and if $Y$ is also locally compact, so is $Z$. If $f$ is a compact map, then $Z$ and $X \cup Y$ are the same. If $X$ and $Y$ are locally compact, Hausdorff semigroups, $(Z, m)$ is a locally compact Hausdorff semigroup provided $m_1$ is a compact map [2].

In this paper we consider the modified Whyburn topology which is coarser than the disjoint union topology, but finer than the Whyburn topology and ask what conditions will insure that $m$ will be continuous.

2. MAIN RESULTS.

Let $W$ denote the disjoint union of $X$ and $Y$ with the modified Whyburn topology; $V$ is open in $W$ iff $V \cap X$ and $V \cap Y$ are open in $X$ and $Y$, respectively, and $f^{-1}(y) \cap V$ is compact for every $y$ in $V \cap Y$. The following notions and facts are due to Stallings [3]. A subset $A$ of $X$ is fiber compact relative to $f$: $X \times Y$ iff $A$ is closed in $X$ and $A \cap f^{-1}(x)$ is compact for every $y \in Y$, and $X$ is locally fiber compact iff every point in $X$ has a neighborhood with a fiber compact closure. Fiber compact subsets of $X$ are closed in $W$ and $W$ is Hausdorff if $X$ is locally fiber compact. If $Y$ is first countable, then $Z$ and $W$
are the same iff \( f \) is closed.

The proof given in [2] that \( m \) is a continuous operation on \( Z \) did not use the assumption that \( m^{-1}_1(K) \) is compact for every compact \( K \) in \( X \), but used an equivalent condition instead. The appropriate generalization of that condition for \( W \) is:

**CONDITION 1.** For every fiber compact \( K_1 \) in \( X \), there is a fiber compact \( K_2 \) in \( X \) such that for all \( x, y \in X \), if \( m_1(x,y) \in K_1 \), then \( x \in K_2 \) and \( y \in K_2 \).

This condition is equivalent to: \( p_i^{-1}(m^{-1}_1(K)) \), \( i = 1,2 \), are fiber compact for each fiber compact \( K \) in \( X \), where \( p_1 \) and \( p_2 \) are the projections on \( X \times X \).

**THEOREM 1.** If \( X \) is locally fiber compact, \( Y \) is regular and \( m_1 \) satisfies Condition 1, then \( m \) is continuous and hence \( W \) is a Hausdorff semigroup.

**PROOF.** The argument is similar to the one given for \( Z \). We will show continuity at a point \((x,y)\) where \( x \in X \) and \( y \in Y \). Let \( w = m(x,y) = m_2(f(x),y) \). Let \( V \) be an open set in \( W \) containing \( w \). Since \( Y \) is regular, there is a \( Y \)-open set \( U \) containing \( y \) such that \( \overline{U} \subset Y \cap V \). Since \( m_2 \) is continuous, there are \( Y \)-open neighborhoods \( U_1 \) and \( U_2 \) of \( f(x) \) and \( y \), respectively, such that \( m_2(U_1 \times U_2) \subset U \cap V \). Then \( V_1 = f^{-1}(U_1) \cup U_2 \), \( i = 1,2 \), are \( W \)-open neighborhoods of \( x \) and \( y \), respectively. Since \( f^{-1}(\overline{U}) \cap V \) is fiber compact, Condition 1 guarantees the existence of a fiber compact \( K \) in \( X \) such that if \( m_1(x,y) \) are in \( f^{-1}(\overline{U}) \) \( - V \), then \( x \) and \( y \) are in \( K \). Since \( K \) is fiber compact, \( K \) is closed in \( W \) and so \( K \times K \) is closed in \( W \times W \). Hence \( V_1 \times V_2 - K \times K \) is an open set containing \((x,y)\) and a calculation shows that \( m \) maps \( V_1 \times V_2 - K \times K \) into \( V \).

Let \( X = (0,1) \times [0,1] \), \( Y = [0,1] \) and \( f: X \rightarrow Y \) by \( f(x,y) = y \). If \( X \) and \( Y \) have the usual multiplications, then \( Z \) is \([0,1] \times [0,1]\) with the usual multiplication. However, the multiplication is not continuous on \( W \) since \( \{(\frac{1}{n},1)\} \rightarrow 1 \) and \( \{(1,1-\frac{1}{n})\} \rightarrow (1,1) \) in \( W \) but \( \{(\frac{1}{n},1-\frac{1}{n})\} \) does not converge since it is a fiber compact set in \( X \) and hence closed in \( W \).

If the multiplication on \( X \) is changed to be the usual multiplication in the first factor and the zero multiplication in the second and if \( Y \) is given the zero multiplication, then the conditions of Theorem 1 are satisfied. Since \( f \) is not a closed map, \( W \) is not the same as \( Z \). Hence \( W \) is a Hausdorff semigroup topologically different from \([0,1] \times [0,1]\).

These examples illustrate how difficult it is to have \( m \) continuous on \( W \). In fact, we have:

**THEOREM 2.** Suppose \( X \) is connected and for each \( y \) in \( Y \), \( f^{-1}(y) \) is not compact. If \((W,m)\) is a first countable, Hausdorff semigroup, then \( Y \) has the zero multiplication.

**PROOF.** Let \( t, y \in Y \) and let \( z = m_2(t,y) \). Let \( A = \{x \in X \mid m(x,y) = z\} \). Since \( f^{-1}(t) \subset A \), \( A \neq \emptyset \). Also \( A \) is closed in \( X \) since \( m(A,y) = z \) implies that \( m(A,y) = z \). Since \( f^{-1}(y) \) is not compact, \( y \) is a limit point of \( f^{-1}(y) \) in \( W \) and so there is a sequence \( \{y_i\} \) in \( f^{-1}(y) \) converging to \( y \) in \( W \). Let \( x \in A \) and \( \{V_i\} \) be a countable neighborhood basis at \( x \). If we assume that no \( V_i \) is contained in \( A \), we can find a sequence \( \{x_i\} \) which converges to \( x \) such that
m(x_i,y) \neq z$. Hence $m_1(x_i,y_i)$ is not in $f^{-1}(z)$ for all $i$, but $\{m_1(x_i,y_i)\}$ converges to $z$. Thus the set $B = \{m_1(x_i,y_i)\}$ is closed in $X$. For any $w \in Y$, $f^{-1}(w) \cap B$ is finite because otherwise $B$ will have a convergent subsequence in the compact set $\{w\} \cup f^{-1}(w)$. This means that $B$ is fiber compact and $W - B$ is a neighborhood of $z$ which contradicts the fact that $\{m_1(x_i,y_i)\}$ converges to $z$. Thus $A$ is open and must equal $X$ since $X$ is connected. All of this yields $m_2(Y,y) = z$. Let $t',y' \in Y$ and let $z' = m_2(t',y')$. The argument above will give that $m_2(t',Y) = z'$. Hence $z = z'$ and $Y$ has the zero multiplication.

REFERENCES

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