COMPATIBLE MAPPINGS AND COMMON FIXED POINTS (2)

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ABSTRACT. A common fixed point theorem of S.L. and S.P. Singh is generalized by weakening commutativity hypotheses and by increasing the number of functions involved.

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1. INTRODUCTION.

In [1] the concept of compatible mappings was introduced as a generalization of commuting maps ($fg=gf$). The utility of compatibility in the context of fixed point theory was demonstrated by extending a theorem of Park and Bae [2]

The purpose of this note is to further emulate the compatible map concept. We extend the following strong result of S.L. Singh and S.P. Singh [3] by employing compatible maps in lieu of commuting maps, and by using four functions as opposed to three.

THEOREM.1.1. Let $P, Q,$ and T be self maps of a complete metric space (X, d) such that $PT=TP, QT=TQ,$ and $P(X) \cup Q(X) \subset T(X)$. If T is continuous and there exists $r \in (0, 1)$ such that $d(Px, Qy) \leq r \max \{d(Tx, Ty), d(Px, Tx), d(Qy, Ty), \frac{1}{2}(d(Px, Ty) + d(Qy, Tx))\}$ for all x, y in X ,
(1.1)
Then $P, Q,$ and T have a unique common fixed point.

2. PRELIMINARIES.

The following definition was given in [1].

Definition 2.1. Self maps f and g of a metric space (X, d) are compatible iff $\lim_n d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X .

Thus, if $d(fgx, gfx) \rightarrow 0$ as $d(fx, gx) \rightarrow 0$, f and g are compatible. For example, suppose that $fx = x^2$ and $gx = 2x^2$ for x in R , the set of reals. f and g are not commutative, but
 $|fx - gx| = x^2 \rightarrow 0$ iff $x \rightarrow 0$ and $|fgx - gfx| = 2x^4 \rightarrow 0$ if $x \rightarrow 0$
so that f and g are compatible on R with the usual metric.

Now maps which commute are clearly compatible, but the converse is false. In fact, compatible maps need not be weakly commutative. Sessa [4] defined self maps f and g of a metric space (X, d) to be a weakly commuting pair iff $d(gfx, fgx) \leq d(fx, gx)$ for x in X . If f and g are weakly commutative they are obviously compatible, but the converse is false as the above example shows (e.g., let $x=1$.). See [1] for other examples of compatible pairs which are not weakly commutative and hence not commuting pairs.

3. MAIN RESULTS.

LEMMA 3.1 . Let A, B, S , and T be self maps of a metric space (X, d) such that $A(X) \subset T(X)$ and $B(X) \subset S(X)$, and let $x_0 \in X$. If $r \in (0, 1)$ such that $d(Ax, By) \leq r \max(M_{xy})$ for $x, y \in X$, where

$$M_{xy} = \{ d(Ax, Sx), d(By, Ty), d(Sx, Ty), \frac{1}{2}(d(Ax, Ty) + d(By, Sx)) \} , \quad (3.1)$$

then there is a Cauchy sequence $\{y_n\}$ in X beginning at x_0 and defined by $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n \in \mathbb{N}$, the set of positive integers.

PROOF. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, we can choose x_1, x_2 in X such that $y_1 = Tx_1 = Ax_0$ and $y_2 = Sx_2 = Bx_1$. In general, we can choose x_{2n-1}, x_{2n} in X such that

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \text{ and } y_{2n} = Sx_{2n} = Bx_{2n-1} \quad (3.2)$$

Thus the indicated sequence $\{y_n\}$ exists.

To see that $\{y_n\}$ is Cauchy, note that (3.1) and (3.2) imply that $d(Tx_{2n+1}, Sx_{2n+2}) = d(Ax_{2n}, Bx_{2n+1}) \leq r \max(M_n)$ where $M_n = \{ d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(Sx_{2n}, Tx_{2n+1}), \frac{1}{2}(d(Ax_{2n}, Tx_{2n+1}) + d(Bx_{2n+1}, Sx_{2n})) \}$. Then by (3.2) $M_n = \{ d(Tx_{2n+1}, Sx_{2n}), d(Sx_{2n+2}, Tx_{2n+1}), \frac{1}{2}d(Sx_{2n+2}, Sx_{2n}) \}$. But $\frac{1}{2}d(Sx_{2n+2}, Sx_{2n}) \leq \frac{1}{2}(d(Sx_{2n+2}, Tx_{2n+1}) + d(Tx_{2n+1}, Sx_{2n})) \leq \max \{ d(Sx_{2n+2}, Tx_{2n+1}), d(Tx_{2n+1}, Sx_{2n}) \}$ since the larger of two numbers is greater or equal to their average. So we have $\max(M_n) = \max \{ d(Sx_{2n+2}, Tx_{2n+1}), d(Tx_{2n+1}, Sx_{2n}) \}$ with $d(Tx_{2n+1}, Sx_{2n+2}) \leq r \max(M_n)$. But if $d(Tx_{2n+1}, Sx_{2n+2}) \leq r d(Tx_{2n+1}, Sx_{2n+2})$, $d(Tx_{2n+1}, Sx_{2n+2}) = 0$, since $r \in (0, 1)$; thus $\max(M_n) = d(Tx_{2n+1}, Sx_{2n})$ and we conclude $d(Tx_{2n+1}, Sx_{2n+2}) \leq r d(Tx_{2n+1}, Sx_{2n})$.

Similarly, $d(Tx_{2n+3}, Sx_{2n+2}) \leq r d(Sx_{2n+2}, Tx_{2n+1})$.

Consequently, (3.2) implies that $d(y_{m+1}, y_m) \leq r d(y_m, y_{m-1})$ for m even or odd.

This last inequality implies that $\{y_m\}$ is Cauchy, as desired. /

We shall also need the following simple result from [1] (Proposition 2.2(2a)).

PROPOSITION 3.1. If f and g are compatible self maps of a metric space (X, d) and $\lim_n f x_n = \lim_n g x_n = t$ for some t in X , then $\lim_n g f x_n = f t$ if f is continuous.

We can now state and prove our generalization of Theorem 1.1.

THEOREM 3.1. Let A, B, S and T be self maps of a complete metric space (X, d) . Suppose that S and T are continuous, the pairs A, S and B, T are compatible, and that $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If there exists $r \in (0, 1)$ such that $d(Ax, By) \leq r \max(M_{xy})$ for x, y in X , where

$$M_{xy} = \{d(Ax, Sx), d(By, Ty), d(Sx, Ty), \frac{1}{2}(d(Ax, Ty) + d(By, Sx))\} \quad (3.3)$$

then there is a unique point z in X such that $z = Az = Bz = Sz = Tz$.

PROOF. By the Lemma 3.1. there is a sequence $\{x_n\}$ in X such that $Sx_{2n} = Bx_{2n-1} = y_{2n}$, $Tx_{2n-1} = Ax_{2n-2} = y_{2n-1}$, and such that the sequence $\{y_m\}$ is Cauchy. Since (X, d) is complete $\{y_m\}$ converges to a point z in X . Consequently, the subsequences

$$\{Ax_{2n}\}, \{Sx_{2n}\}, \{Tx_{2n-1}\}, \{Bx_{2n-1}\} \text{ converge to } z. \quad (3.4)$$

Since A and S are compatible and B and T are compatible, the continuity of S and T , (3.4), and Proposition 3.1. imply

$$TTx_{2n-1}, BTx_{2n-1} \rightarrow Tz \text{ and } SSx_{2n}, ASx_{2n} \rightarrow Sz. \quad (3.5)$$

Then (3.3) implies $d(Sz, Tz) = \lim_n d(ASx_{2n}, BTx_{2n-1}) \leq r \max(\lim_n M_n)$ where $M_n = \{d(ASx_{2n}, SSx_{2n}), d(BTx_{2n-1}, TTx_{2n-1}), d(SSx_{2n}, TTx_{2n-1}), \frac{1}{2}(d(ASx_{2n}, TTx_{2n-1}) + d(SSx_{2n}, BTx_{2n-1}))\}$. By (3.5), $\lim_n M_n = \{0, 0, d(Sz, Tz), \frac{1}{2}(d(Sz, Tz) + d(Sz, Tz))\}$, so that $d(Sz, Tz) \leq r d(Sz, Tz)$. Since $0 < r < 1$, $Sz = Tz$.

Also, $d(Az, Tz) = \lim_n d(Az, BTx_{2n-1}) \leq r \max(\lim_n M_n)$ where $M_n = \{d(Az, Sz), d(BTx_{2n-1}, TTx_{2n-1}), d(Sz, TTx_{2n-1}), \frac{1}{2}(d(Sz, BTx_{2n-1}) + d(Az, TTx_{2n-1}))\}$. Since $Sz = Tz$, (3.5) yields: $\lim_n M_n = \{d(Az, Tz), 0, 0, \frac{1}{2}d(Az, Tz)\}$; therefore, $d(Az, Tz) \leq r d(Az, Tz)$ from which (as above) we infer $Az = Tz (= Sz)$. But if we use this last stated equality in (3.3) with $x=y=z$, we obtain:

$$Az = Bz = Sz = Tz. \quad (3.6)$$

In fact, z is a common fixed point of A, B, S , and T . For (3.3) and (3.4) yield: $d(z, Bz) = \lim_n d(Ax_{2n}, Bz) \leq r \max(\lim_n M_n)$ with $M_n = \{d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), d(Sx_{2n}, Tz), \frac{1}{2}(d(Ax_{2n}, Tz) + d(Bz, Sx_{2n}))\}$. Then $\lim_n M_n = \{0, 0, d(z, Tz), \frac{1}{2}(d(z, Bz) + d(Bz, z))\}$ by (3.4) and (3.6). We thus

obtain $d(z, Bz) \leq r d(z, Bz)$, and we conclude that $z = Bz = Az = Sz = Tz$. That z is the only common fixed point of A, B, S , and T follows easily from (3.3)./

We conclude with an example of four functions which satisfy the hypothesis of Theorem 3.1., no three of which satisfy the hypothesis of Theorem 1.1..

EXAMPLE 3.1. Let $X = [1, \infty)$ and $d(x, y) = |x - y|$ for $x, y \in X$. Define $Ax = x^3$, $Bx = x^2$, $Sx = 2x^6 - 1$ and $Tx = 2x^4 - 1$ for x in X . The functions are all continuous and satisfy $A(X) = B(X) = S(X) = T(X) = X$. Moreover, $|Sx - Ax| = |2x^7 + 1| |x^3 - 1| \rightarrow 0$ iff $x \rightarrow 1$ since $x \geq 1$, and $|ASx - SAx| = 6x^6(x^6 - 1)^2 \rightarrow 0$ iff $x \rightarrow 1$ since $x \geq 1$. Thus, $d(Ax, Sz) \rightarrow 0$ only if $x \rightarrow 1$, in which instance $d(ASx, Sax) \rightarrow 0$. So A and S are compatible; but they are not a weakly commuting and hence not a commuting pair (Let $x = 2$). Similarly, T and B are compatible, since $|Tx - Bx| = (2x^2 + 1) |x^2 - 1| \rightarrow 0$ iff $x \rightarrow 1$ ($x \geq 1$) and $|TBx - BTx| = 2(x^4 - 1)^2 \rightarrow 0$ iff $x \rightarrow 1$ ($x \geq 1$). Finally, $|Sx - Ty| = 2 |x^7 - y^2| |x^3 + y^2| \geq 2 |Ax - By|$ for $x, y \geq 1$; therefore, $|Ax - By| \leq \frac{1}{2} |Sx - Ty| \leq \frac{1}{2} \max(Mxy)$ for x, y in X . Hence (3.3), and thus the hypothesis of Theorem 3.1., is satisfied. Observe also that no one of A, B, S , or T commutes with any two of the remaining three functions.

Of course, common fixed point theorems other than Theorem 1.1 follow from Theorem 3.1.. See, for example, Corollary 3.2 of [1] which in turn has Theorem 1. of [5] as a corollary.

REFERENCES

1. Jungck, G. Compatible mappings and common fixed points Internat. J. Math. Math. Sci. 9(1936)
2. Park, S. and Bae, Jong Sook Extensions of a fixed point theorem of Meir and Keeler. Ark. Mat. 19(1981) 223-228
3. Singh, S.L. and Singh, S.P. A fixed point theorem. Indian J. Pure Appl. Math. 11(1980) 1584-1586.
4. Sessa, S. On a weak commutativity condition in fixed point considerations, Publ. Inst. Math. 32(46)(1982) 149-153.
5. Hadzic, Olga Common fixed point theorems for family of mappings in complete metric spaces. Math. Japonica 29(1984) 127-134.