QUADRATIC SUBFIELDS OF QUARTIC EXTENSIONS OF LOCAL FIELDS

JOE REPKA

Mathematics Department
University of Toronto
Toronto, Ontario
CANADA M5S 1A1

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ABSTRACT. We show that any quartic extension of a local field of odd residue characteristic must contain an intermediate field. A consequence of this is that local fields of odd residue characteristic do not have extensions with Galois group A_4 or S_4 . Counterexamples are given for even residue characteristic.

KEY WORDS AND PHRASES. Local field, quartic extension, endoscopic group.

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1. INTRODUCTION.

In Section 2, a simple application of local class field theory proves the existence of intermediate fields for quartic extensions of local fields with odd residue characteristic. This immediately implies the non-existence of Galois extensions of type A_4 or S_4 over such fields.

In Section 3, examples are given of A_4 and S_4 extensions of fields with even residue characteristic, and of a quartic extension with no intermediate field.

In Section 4, the results of Section 2 are used to show that the splitting field of an irreducible quartic polynomial over a local field must have degree 4 or 8, provided the residue characteristic is odd. The implications of the results of Section 2 and Section 3 for the theory of endoscopic groups are also discussed.

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2. EXISTENCE OF INTERMEDIATE EXTENSIONS.

Let F be a non-archimedean local field. Let $\mathfrak{o} = \mathfrak{o}_F$ and $\mathfrak{p} = \mathfrak{p}_F$, respectively, be the ring of integers of F and its prime ideal.

THEOREM 2.1. Suppose the residue characteristic of F is odd, and E/F is a quartic extension (i.e. $[E:F] = 4$). Then there must be an intermediate field K , i.e. $E \supset K \supset F$, $[E:K] = [K:F] = 2$.

PROOF: If E/F is unramified, the result is obvious. If the ramification index of E/F is $e = 2$, then we must have $f = 2$ and, by Corollary 4 to Theorem 7 of chapter I, Section 4 of Weil [1], there is an unramified quadratic intermediate field.

Now suppose $e = 4$, so $f = 1$. Any unit in E is of the form $u + p$, with $u \in \mathcal{O}_F^\times$ and $p \in \mathfrak{p}_E$. The norm of such an element is $u^4 + p'$, with $p' \in \mathfrak{p}_E \cap F = \mathfrak{p}_F$. So by Hensel's Lemma the only units contained in the image of $N_{E/F}$ are fourth powers. In particular, $N_{E/F}$ is not surjective, so Corollary 1 to Theorem 4 of chapter XII, Section 3 of Weil [1] proves the theorem.

Translating this into the corresponding result on Galois groups, we obtain the following equivalent formulation ...

THEOREM 2.2. If F has odd residue characteristic, there cannot be a Galois extension E/F whose Galois group is isomorphic to A_4 or S_4 .

PROOF: A_4 contains subgroups of index 4 (the cyclic group generated by any 3-cycle), none of which is properly contained in any proper subgroup (such a proper subgroup, if it existed, would be of order 6 and index 2, hence normal, hence would contain all 3-cycles, of which there are 8).

An S_4 -extension of F would be an A_4 -extension of a quadratic extension of F .

3. COUNTEREXAMPLE FOR RESIDUE CHARACTERISTIC 2.

Let $F = \mathbb{Q}_2$ and consider the Eisenstein polynomial $\Phi(X) = X^4 - 2X - 2 \in F[X]$. Let E be the splitting field of $\Phi(X)$; we shall show that $\text{Gal}(E/F) = S_4$ and $\text{Gal}(E/K) = A_4$, where $K = \mathbb{Q}_2(\sqrt{3})$. In the process we shall find a quartic extension L/F with no intermediate field.

Let α be a root of $\Phi(X)$, and let $L = F(\alpha)$.

LEMMA 3.1. The norm $N_{L/F}$ is surjective.

PROOF: Notice that $N(\alpha+1) = \Phi(-1) = 1$, $N(\alpha-1) = \Phi(1) = -3$. Also the characteristic polynomial of α^3 is $\Phi_3(X) = X^4 - 6X^3 + 12X^2 - 8X - 8$, so $N(\alpha^3+1) = \Phi_3(-1) = 19$. If $N = N_{L/F}$ were not surjective, its image would be contained in the image of the norm map from some ramified quadratic extension of F . Such an image contains exactly two of the four cosets of $\mathcal{O}^\times$ modulo $(\mathcal{O}^\times)^2$. We have just shown $N_{L/F}$ contains the three cosets containing 1, -3, and 19.

In particular (by Corollary 1 to Theorem 4 of chapter XII, Section 3 of Weil [1]), L/F is a quartic extension with no intermediate field.

Factoring the polynomial $\Phi(X)$ over L , we see that $\Phi(X) = (X-\alpha)\Psi(X)$, where $\Psi(X) = X^3 + \alpha X^2 + \alpha^2 X + (\alpha^3 - 2)$.

PROPOSITION 3.2. $\Psi(X)$ is irreducible over L .

PROOF: If all roots of $\Psi(X)$ were in L , then $L = E$ would be Galois, in contradiction of Lemma 3.1. The only other way for $\Psi(X)$ to be reducible would be for exactly one root, α' say, to be in L . In this case,

$F(\alpha')$ would be a quartic extension of F contained in L , hence $F(\alpha') = F(\alpha) = L$.

Let $\sigma \in \text{Gal}(E/F)$ be such that $\sigma(\alpha) = \alpha'$. Then $\sigma(F(\alpha)) = F(\alpha')$, and $\alpha' \in F(\alpha)$ implies that $\sigma(\alpha') \in F(\alpha') = F(\alpha) = L$. Since $\sigma(\alpha') \neq \alpha'$, $\sigma(\alpha')$ must equal the only other conjugate of α' in L , i.e. $\sigma(\alpha') = \alpha$. Hence the fixed field L^σ contains $\alpha + \alpha'$ and $\alpha\alpha'$, so $(X-\alpha)(X-\alpha') = X^2 - (\alpha+\alpha')X + \alpha\alpha' \in L^\sigma[X]$, which shows that α is quadratic over L^σ . So $[L:L^\sigma] = [L^\sigma:F] = 2$. This also contradicts Lemma 3.1.

So E is the splitting field of $\Psi(X)$ over L , and $\text{Gal}(E/L)$ is either A_3 or S_3 .

Now $\Psi(X) = X^3 + \alpha X^2 + \alpha^2 X + \alpha^3 - 2 = X'^3 + (2/3)\alpha^2 X' + (20/27)\alpha^3 - 2$, where $X' = X + 2/3$. Hence the discriminant of $\Psi(X)$ is $27((20/27)\alpha^3 - 2)^2 - 4((2/3)\alpha^2)^3 = 4.27 + (368/27)\alpha^6 - 80\alpha^3 \equiv 4.9.3 \pmod{1+4p_L}$.

Since $4.9(1+4p_L) \subset (L^\times)^2$, the discriminant of $\Psi(X)$ is a square in L if and only if 3 is.

LEMMA 3.3. The element 3 is not a square in L .

PROOF: If 3 were a square, truncation of its square root would give an element of the form $x = 1 + a\alpha + b\alpha^2 + c\alpha^3$, with a, b , and c each equal to 0 or 1 and so that $3 - x^2 \in 4p_L$. A trivial computation shows that this is impossible.

Accordingly $\text{Gal}(E/L) = S_3$, $\text{Gal}(E/F) = S_4$, and $\text{Gal}(E/K) = A_4$, where $K = F(\sqrt{3})$.

4. APPLICATIONS.

1. The splitting field of a quartic polynomial over a local field is severely constrained by the results of Section 2.

THEOREM 4.1. Let F be a local field with odd residue characteristic. Let $f(X) \in F[X]$ be an irreducible polynomial with $\deg f(X) = 4$. Let E be the splitting field of $f(X)$ over F . Then $[E:F] = 4$ or 8.

PROOF: $\text{Gal}(E/F)$ is a subgroup of S_4 . But by Theorem 2.2 it cannot be S_4 or A_4 . Since $4 \mid [E:F]$, the only possibilities are 4 or 8.

The polynomial $\Phi(X)$ of Section 3 gives a counterexample to this result when the residue characteristic is 2. Theorem 4.1 is clearly equivalent to Theorem 2.2 (and hence to Theorem 2.1).

2. If F is a local field, let $G = \text{SL}(4, F)$, and let T be an elliptic torus in G . To T is associated a quartic extension E/F so that the centralizer of T in $\text{GL}(4, F)$ is isomorphic to $E^\times$, and T itself is isomorphic to $E_1^\times = \{x \in E^\times; N_{E/F}(x) = 1\}$.

The theory of endoscopic groups (cf. Langlands [2], Shelstad [3]) associates to G and T some other groups, among which the most interesting are constructed as follows: let $E \supsetneq K \supsetneq F$ and let $G' = \{g \in \text{GL}(2, K) : N_{K/F}(\det g) = 1\}$. In G' it is possible to find an

elliptic torus T' associated to the quadratic extension E/K , and there is an isomorphism between T and T' . The hope is to simplify calculations with orbital integrals over the G -conjugacy class of $t \in T$ by comparing them with orbital integrals over the G' -conjugacy class of the corresponding $t' \in T'$.

The example of Section 3 shows that this approach will not apply for certain tori when the residue characteristic is 2; happily, for these tori the ordinary orbital integrals are invariant under stable conjugacy, so the problem does not arise. The results of Sections 2 encourage optimism in the case of odd residue characteristic.

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