MAXIMAL SUBALGEBRA OF DOUGLAS ALGEBRA

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ABSTRACT. When \( q \) is an interpolating Blaschke product, we find necessary and sufficient conditions for a subalgebra \( B \) of \( H^\infty[q] \) to be a maximal subalgebra in terms of the nonanalytic points of the noninvertible interpolating Blaschke products in \( B \). If the set \( M(B) \cap Z(q) \) is not open in \( Z(q) \), we also find a condition that guarantees the existence of a factor \( q_0 \) of \( q \) in \( H^\infty \) such that \( B \) is maximal in \( H^\infty[q] \). We also give conditions that show when two arbitrary Douglas algebras \( A \) and \( B \), with \( A \subseteq B \) have property that \( A \) is maximal in \( B \).

KEY WORDS AND PHRASES. Maximal subalgebra, Douglas algebra, interpolating sequence, sparse sequence, Blaschke product, inner functions, open and closed subset, nonanalytic points, support set, \( Q\)-\( C \) level sets.


1. INTRODUCTION.

Let \( D \) be the open unit disk in the complex plane and \( T \) be its boundary. Let \( L^\infty \) be the space of essentially measurable functions on \( T \) with respect to the Lebesgue measure. By \( H^\infty \) we mean the family of all bounded analytic functions in \( D \). Via identification with boundary functions, \( H^\infty \) can be considered as a uniformly closed subalgebra of \( L^\infty \). A uniformly closed subalgebra \( B \) between \( H^\infty \) and \( L^\infty \) is called a Douglas algebra. If we let \( C \) be the family of continuous functions on \( T \), then it is well known that \( H^\infty + C \) is the smallest Douglas algebra containing \( H^\infty \) properly. For any Douglas algebra \( B \), we denote by \( M(B) \) the space of nonzero multiplicative linear functionals on \( B \), that is, the set of all maximal ideals in \( B \). An algebra \( B_0 \) is said to be a maximal subalgebra of \( B \), if \( B_1 \) is another algebra with the property that \( B_0 \subseteq B_1 \subseteq B \), then either \( B_1 = B_0 \) or \( B_1 = B \).

An interpolating sequence \( \{z_n\}_{n=1}^\infty \) is a sequence in \( D \) with the property that for any bounded sequence of complex numbers \( \{\lambda_n\}_{n=1}^\infty \), there exists \( f \) in \( H^\infty \) such that \( f(z_n) = \lambda_n \) for all \( n \). A well-known condition states that a sequence \( \{z_n\}_{n=1}^\infty \) is interpolating if and only if

\[
\inf_{n \neq m} \frac{|z_m - z_n|}{|1 - \frac{z_n}{z_m}|} = \delta > 0. \tag{1.1}
\]
A Blaschke product
\[ q(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \left( \frac{z - z_n}{1 - \overline{z_n}z} \right) \]  

is called an interpolating Blaschke product if its zero set \( \{z_n\}_{n=1}^{\infty} \) is an interpolating sequence (\( |z_n|/z_n = 1 \) is understood whenever \( z_n = 0 \)). A sequence \( \{z_n\}_{n=1}^{\infty} \) is said to be sparse if it is an interpolating sequence and
\[ \lim_{n \to \infty} \prod_{n \neq m}^{\infty} \frac{|z_m - z_n|}{1 - \overline{z_n}z_m} = 1. \]  

For a function \( q \) in \( H^C \), we let \( Z(q) = \{m \in M(H^C): q(m) = 0\} \) be the zero set of \( q \) in \( M(H^C) \). An inner function is a function in \( H^C \) of modulus 1 almost everywhere on \( T \). We denote by \( H^C \) the Douglas algebra generated by \( H^C \) and the complex conjugate of the inner function \( b \).

We put \( X = M(L) \). Then \( X \) is the Silow boundary for every Douglas algebra. For a point in \( M(H^C) \), we denote by \( \mu_x \) the representing measure on \( X \) for \( x \) and by \( \text{supp} \mu_x \) the support set for \( \mu_x \). For a function \( q \) in \( L^C \) (in particular if \( q \) is an interpolating Blaschke product), we put \( N(q) \) the closure of the union set of \( \text{supp} \mu_x \) such that \( x \in M(H^C) \) and \( \overline{q}\mu_x \mu_x^* \). Roughly speaking, \( N(q) \) is the set of nonanalytic points of \( q \). Set \( QC = H^C \cap H^C \) and for \( x_0 \) in \( X \), let \( Q_{x_0} = \{x \in X: f(x) = f(x_0) \text{ for } f \in QC\} \). \( Q_{x_0} \) is called the QC-level set for \( x_0 \) [9]. For an inner function \( q \), K. Izuchi has shown the following [5, Theorem 1(i)].

**Theorem 1.** If \( q \) is an inner function that is not a finite Blaschke product, then,
\[ N(q) = \bigcup \{Q_x; x \in Z(q)\}. \]  

In particular, the right side of \( 1.4 \) is a closed set. Now assume that \( q \) is an interpolating Blaschke product, and let \( B \) be a Douglas algebra contained in \( H^C[q] \). We will always assume that \( M(B)^* \cap Z(q) \) is not an open set in \( Z(q) \), for Izuchi has shown [6] that if \( B \) is a maximal subalgebra of \( H^C[q] \), then \( M(B)^* \cap Z(q) \) is not open in \( Z(q) \). We will give answers to the following two questions. When is \( B \) a maximal subalgebra of \( H^C[q] \) or when is there a factor \( q_0 \) of \( q \) in \( H^C \) such that \( B \) is maximal in \( H^C[q_0] \)? These answers will be in terms of the nonanalytic points of \( q \) and the invertible inner functions of \( H^C[q] \) that are not invertible in \( B \).

For a Douglas algebra \( B \), we denote by \( N(B) \) the closure of \( \{\text{supp} \mu_x; x \in M(H^C)/M(B)\} \). In particular \( N(H^C[q]) = N(q) \). In general if \( A \) and \( B \) are Douglas algebras such that
A ⊆ B, we put \( N_A(B) \) = the closure of \( \bigcup \{ \supp x : x \in M(A)/M(B) \} \) and for any inner function \( b \), \( N_A(b) \) = the closure of \( \bigcup \{ \supp x : x \in M(A), |b(x)| < 1 \} \).

It is shown in [7, Corollary 2.5] that if \( B \subseteq H^\infty[q] \), then \( N(B) \subseteq N(q) \), and it is not hard to show that \( N(q)/N(B) \supseteq N_B(q) \) (in a sense the set \( N_B(q) \) is generated by the nonanalytic points \( M(B)/M(H^\infty[q]) \subseteq M(H^\infty + C)/M(H^\infty[q]) \)).

2. OUR MAIN RESULT.

We'll need the following lemma. It shows how small \( M(B)/M(H^\infty[q]) \) must be if \( B \) is to be a maximal subalgebra of \( H^\infty[q] \). Let \( Q = \{ b : b \) is an interpolating Blaschke product \} with \( b \in H^\infty[q] \), and \( Q(B) = \{ b \in Q : b \notin B \} \).

**Lemma 1.** Let \( q \) be an interpolating Blaschke product and \( B \) be a Douglas algebra contained in \( H^\infty[q] \). Suppose for all \( b \in Q(B) \), we have that \( N_B(q) \subseteq N_B(b) \). Then \( B \) is a maximal subalgebra of \( H^\infty[q] \).

**Proof.** It suffices to show that if \( b \in Q(B) \), then \( B[b] = H^\infty[q] \). Hence the only Douglas algebra between \( B \) and \( H^\infty[q] \) that contains \( B \) properly is \( H^\infty[q] \). We show that \( M(B[b]) \subseteq M(H^\infty[q]) \). Now \( M(B[b]) = \{ m \in M(B) : |b(m)| = 1 \} \). It suffices to show that if \( m \notin M(H^\infty[q]) \), then \( m \notin M(B[b]) \). Let \( m \in M(B) \) such that \( m \notin M(H^\infty[q]) \). Then \( q \big| \supp m \subseteq H^\infty \) and since \( N_B(q) \subseteq N_B(b) \), we have that \( b \big| \supp m \subseteq H^\infty \). Thus \( |b(m)| < 1 \) and we get \( m \notin M(B[b]) \). This shows that \( M(B[b]) \subseteq M(H^\infty[q]) \), and \( B \) is maximal in \( H^\infty[q] \).

Using Theorem 1 above, it is not hard to show directly that \( N(B[b]) = N(q) \). However, by Proposition 4.1 of [7], this condition is not sufficient.

We let \( E = N_B(q) \). This can be a very complicated set. For example, it can contain \( \supp x \) where \( x \) belongs to a trivial Gleason part or a Gleason part where \( |q| < 1 \), but yet \( q \neq 0 \) on this part [see 3]. So for \( B \) to be maximal in \( H^\infty[q] \), \( E \) must be as simple as possible. To see how simple, we set \( \Lambda(B) = \{ b \in Q(B) : b \subseteq H^\infty[b] \} \) and \( \Lambda^*(B) = \{ a \in \Lambda(B) : a \notin \Lambda(B) \} \). Now let \( E^* = \cap b \in \Lambda(B) N(b) \), \( E^{**} = \cap b_0 \in Q(B) N(b_0) \), \( E^*_0 = E^* \cap E \) and \( E^{**}_0 = E^{**} \cap E \). Note that if \( E^{**}_0 = \emptyset \), then there are interpolating Blaschke products \( a_0 \) and \( a_1 \) in \( \Lambda^*(B) \) such that \( N_B(q) \cap N(a_0) \cap N(a_1) = \emptyset \). Thus we get \( B \subseteq B[a_0] \subseteq H^\infty[q] \). To see this, just note that both \( N_B(q) \cap N(a_0) \) \( \cap N(a_1) \) \( \cap \emptyset \) and \( N_B(q) \cap N(a_1) \) \( \cap \emptyset \) since \( a_0 \) and \( a_1 \) belong to \( \Lambda^*(B) \). Since their intersection is empty, there is an \( x_1 \in M(B) \) such that \( a_0 \big| \supp x_1 \subseteq H^\infty \). Thus \( N_B^{-1}(q) \subseteq N(q) \), which implies that \( B[a_0] \subseteq H^\infty[q] \).

Obviously, \( B \subseteq B[a_0] \), so \( B \) cannot be maximal in \( H^\infty[q] \) unless \( E^{**}_0 \neq \emptyset \). We now state.

**Proposition 1.** Let \( B \) be a Douglas algebra properly contained in \( H^\infty[q] \), and suppose \( E^{**}_0 \neq \emptyset \). Then the following statements are equivalent:

(i) \( N(B) = N(q) \);

(ii) \( B \) is a maximal subalgebra of \( H^\infty[q] \);

(iii) \( E^{**}_0 = E^*_0 = E \);

(iv) \( E^*_0 = N_B(q) \).
PROOF. We prove the following: (i) $\rightarrow$ (ii) $\rightarrow$ (iii) $\rightarrow$ (iv) $\rightarrow$ (ii) $\rightarrow$ (i).

Suppose (i) holds. We will show that $N_B^{-}(q) \subseteq N_B^{-}(b)$ for all $b \in \Omega(B)$. Using Lemma 1, this will prove that $B$ is a maximal subalgebra of $H[w^{-}(q)]$. Let $b \in \Omega(B)$ and consider the Douglas algebra $B[b]$. We have $B \subseteq B[b] \subseteq H[w^{-}(q)]$, hence $N(B) = N(B[b]) = N(q)$. Now $N(q) = N(B) \cup N_B^{-}(q)$, so by the above equality we have that $N_B^{-}(q) \subseteq N(B[b])$. Thus, if $x \in M(B)$ such that $\overline{\text{supp } x} \subseteq H[w^{-}(q)]$, implies that $\overline{\text{supp } x} \subseteq N(B[b])$. Thus $b|_{\overline{\text{supp } x}} \in H[w^{-}(q)]$, and $N_B^{-}(q) \subseteq N_B^{-}(b)$. We have (i) $\rightarrow$ (ii).

Next suppose that (ii) holds. It is clear that $E_0 \subseteq E_0^* \subseteq E$. We must show that $E_0^*|_{E_0^*}$ and $E|_{E_0}$ are empty sets. First we show that $E_0 \subseteq E$ is empty. Suppose not. Then there is an $x \in M(B)$ and a $b_0 \in \Lambda^*(B)$ such that $b_0|_{\overline{\text{supp } x}} \in H[w^{-}(q)]$ and $\overline{\text{supp } x} \subseteq E_0^*$. It is clear by Theorem 1 that $\overline{\text{supp } x} \subseteq N(B_0^*) = \emptyset$. Consider the algebra $B_{[b_0]}$. Since $b_0 \in \Lambda^*(B)$, $E \subseteq B_{[b_0]}$. Since $\overline{\text{supp } x} \subseteq N(q)$ and $\overline{\text{supp } x} \subseteq N(b_0)$, we have that $|b_0(x)| = 1$, so we have $\overline{\text{supp } x} \subseteq N(q)/N_{[b_0]}^{-}(q)$. This implies that $B_{[b_0]} \subseteq H[w^{-}(q)]$, which is a contradiction. So $E_0^* = E_0$.

Now we show that $E/E_0^*$ is empty. Again suppose not. Hence there is a $y \in M(B)$ such that $\overline{\text{supp } y} \subseteq E$, but $\overline{\text{supp } y} \subseteq E_0^*$. There is a $b \in \Lambda(B)$ such that $\overline{\text{supp } y} \subseteq N_B^{-}(b)$. Again this implies that $b|_{\overline{\text{supp } y}} \in H[w^{-}(q)]$, thus we have that $B \not\subseteq B[b]$ (since $b \not\in \Lambda(B)$ and $B[b] \subseteq H[w^{-}(q)]$ (since $\overline{\text{supp } y} \subseteq N(q)/N_{[b]}^{-}(q)$), which is a contradiction. So we get $E_0^* = E$. This shows that (ii) $\rightarrow$ (iii).

It is trivial that if (iii) holds, $E_0^* = E$.

If (iv) holds and $b$ is any interpolating Blaschke product in $\Omega(B)$, then by (iv) $N_B^{-}(q) \subseteq N_B^{-}(b)$ so by Lemma 1, $B$ is a maximal subalgebra of $H[w^{-}(q)]$.

Finally, suppose (ii) holds. We are going to show that $N(B) = N(q)$. Suppose not. Then $N(B) \subset N(q)$. By Theorem 1 there is a Q-C level set $Q$ with $N(B) \cap Q = \emptyset$. Put $B_0 = [H[w^{-}], I; I$ is an interpolating Blaschke product with $\overline{I} \subseteq H[w^{-}]$ and $\overline{I} \subseteq H[w^{-}]$. By Proposition 4.1 of [7], we have $B_0 \subseteq H[w^{-}(q)]$ and $N(B_0) = N(q)$. Since $N(B) \cap Q = \emptyset$, we also have $B \subseteq B_0$ (because $N(B) \subseteq N(B_0)$). This implies that $B$ is not a maximal sub-algebra of $H[w^{-}(q)]$, which is a contradiction. Thus $N(B) = N(q)$.

Now suppose we have that $E_0^* \subseteq E_0 \subseteq E$ ($E_0^* = \emptyset$ is possible). When is there a factor $q_0$ of $q$ in $H[w^{-}]$ such that $B$ is a maximal subalgebra of $H[w^{-}(q_0)]$ ($B = H[w^{-}(q_0)]$ is not possible)? To answer this question, let $\Omega_0 = \{q_0 : q_0 \in H[w^{-}]\}$, and $\Omega_0(B) = \{q_0 \in \Omega_0 : B \subseteq H[w^{-}(q_0)]\}$.

Set $F = \bigcap_{q_0 \in \Omega_0(B)} N(q_0)$. Suppose $F = N(q_0)$ for some factor $q_0$ of $q$ in $H[w^{-}]$. Then $B \subseteq H[w^{-}(q_0)]$. So $q_0$ is our possible candidate. Next, let $\Omega = \{c : c$ is an interpolating Blaschke product with $c \in H[w^{-}(q_0)]\}$,
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\[ \Omega_q(\mathcal{B}) = \Omega_q \cap \Omega(\mathcal{B}), \quad \Lambda_q(\mathcal{B}) = \Omega_q \cap \Lambda(\mathcal{B}), \quad \Lambda^*_q(\mathcal{B}) = \Omega_q \cap \Lambda^*(\mathcal{B}), \]

\[ F_0 = E \cap N(q_0), \quad F^* = F_0 \cap F, \quad F^{**} = \bigcap_{c \in \Omega_q(\mathcal{B})} N(c), \quad F_0^* = F^* \cap F_0, \quad \text{and finally} \]

\[ F_0^{**} = F^{**} \cap F_0. \]

We have the following.

**COROLLARY I.** Let \( q_0 \) be a factor of \( q \) in \( H^\infty \) such that \( F = N(q_0) \) and assume \( F_0^{**} \neq \emptyset \).

If any of the following conditions hold:

(i) \( F_0 = F_0 = F_0 \)

(ii) \( F_0^{**} = N_B(H_0), \) where \( H_0 = \bigcap_{q_0 \in \Omega_q(\mathcal{B})} H^\infty \).

Then \( B \) is a maximal subalgebra of \( H_0 = H^\infty[q_0] \) where \( q_0 \in \Omega_q(\mathcal{B}) \).

The fact that \( F = N(q_0) \) for some \( q_0 \in \Omega_q(\mathcal{B}) \) implies that \( H_0 = H^\infty[q_0] \) and our corollary follows from Proposition 1.

We now consider this question for the general Douglas algebras. Let \( A \) and \( B \) be Douglas algebras such that \( A \subseteq B \) and there is an inner function \( q \) with \( B \subseteq A[q] \).

When this occurs we say that \( A \) is near \( B \). It is well known that if \( B = L^\infty \) and \( A \) is any Douglas algebra properly contained in \( B \), then \( A \) is not near \( B \), that is, \( B \subseteq A[q] \) for any inner function \( q \). In fact \( L^\infty \) is not countably generated over any Douglas algebra \( A \) [10]. So by the results of C. Sundberg [10] any Douglas algebra \( B \) which is countably generated over \( A \) is also near it.

The following result comes from [2, Lemma 5] and gives equivalent conditions for two Douglas algebras to be near each other [see [11, Theorem 1] for a similar result].

**THEOREM 2.** Let \( A \) and \( B \) be Douglas algebras with \( H^\infty + c \subseteq A \subseteq B \) and \( q \) be an inner function. Then the following statements are equivalent:

(i) \( M(A) = Z_A(q) \cup M(B) \)

(ii) \( \phi B \subseteq A \).

where \( Z_A(q) = Z(q) \cap M(A) \).

PROOF. Assume (i) holds; we show that \( \phi B \subseteq A \). Let \( b \) be any interpolating Blaschke product for which \( \overline{b} \) is in \( B \). If \( x \) is in \( Z_A(b) \), we show that \( x \) is also in \( Z_A(q) \). Now \( x \) is in \( M(A) \) and \( b(x) = 0 \) implies that \( x \) is not in \( M(B) \), since \( \overline{b} \) is in \( B \). So by (i) we have that \( x \) must be in \( Z_A(q) \). Thus \( Z_A(b) \subseteq Z_A(q) \), and by Theorem 1 of [4] we have \( \overline{b} \) is in \( A \). Now let \( f \) be any function in \( B \). By the Chang Marshall Theorem [1,8] there is a sequence of functions \( \{h_n\} \) in \( H^\infty \) and a sequence of interpolating Blaschke products \( \{b_n\} \) with \( b_n \neq B \) for all \( n \), such that \( h_n b_n \rightarrow f \).

But \( h_n b_n \rightarrow f \) belongs to \( A \) since \( b_n \in A \) for all \( n \). This proves (ii).
Assume (ii) holds. Let x be in M(A) but not in M(B). Then there is an inner function b which is invertible in B such that |b(x)| < 1. For any positive integer n, the function \( f_n = q^{n} \) is in A, so

\[ |g(x)| = |b(x)|^{n} |f_n(x)| \leq |b(x)|^{n}. \]

Letting \( n \to \infty \) we get \( (x) = 0 \). This proves (i).

Set \( Z_B(q) = M(B) \cap Z_A(q) \) and \( Z_B^*(q) = Z_A(q)/Z_B(q) \); then \( M(A)/M(B) = \bigcup_{x \in Z_B^*(q)} P_x \), since \( M(A) = M(B) \cup Z_A(q) \).

As we have previously done, let \( \Omega(B,A) \) be the set of interpolating Blaschke products b such that \( b \in B \) but \( b \notin A \) and set \( W = \bigcap_{b \in \Omega(B,A)} N_A(b) \). We assume \( W \neq \emptyset \).

Using Proposition 1, Theorem 2 and Lemma 1, we have the following result.

**PROPOSITION 2.** Let A and B be arbitrary Douglas algebras such that A is near B. Then the following statements are equivalent:

(i) \( N_A(B) \subseteq N_A(b) \) for all \( b \in \Omega(B,A) \);

(ii) A is a maximal subalgebra of B;

(iii) \( W^* = N_A(B) \).

**PROOF.** Assume that (i) holds. Since A is near to B, there is an inner function such that \( M(A) = M(B) \cup \bigcup_{x \in Z_B^*(\phi)} P_x \). If we set \( A^* = \bigcup_{x \in Z_B^*(\phi)} P_x \), then it is immediate that

\[ N_A(B) = \text{closure of } \bigcup \{ \text{supp } x : x \in A^* \}. \]

Let \( b \) be any element in \( \Omega(B,A) \). By (i) we have that \( N_A(B) \subseteq N_A(b) \). As in proof of Lemma 1 we have that \( A^*[b] = B \). Thus is a maximal in B.

Assume that (ii) holds, and let \( x \in A^* \). Since A is near B, we have that \( M(A) = M(B) \cup A^* \). If \( b \in \Omega(B,A) \), then by our hypothesis \( A^*[b] = B \), which implies that if \( y \in M(A) \) and \( |b(y)| = 1 \), then \( y \in M(B) \) (since \( M(A[b]) = \{ g \in M(A) : |b(g)| = 1 \} = M(B) \)).

So, if \( \text{supp } x \subseteq N_A(B) \), the \( b|_{\text{supp } x} \) \( H_x|_{\text{supp } x} \), Thus \( N_A(B) \subseteq N_A(b) \) for all \( b \in \Omega(B,A) \).

This implies that \( N_A(B) \subseteq W^* \).

To show what \( W^* \subseteq N_A(B) \), let \( b \in \Omega(B,A) \). Hence \( b \in B \); therefore we have

\[
N_A(b) = \text{closure of } \bigcup \{ \text{supp } x : x \in M(A), |b(x)| < 1 \}
= \text{closure of } \bigcup \{ \text{supp } x : x \in M(A)/M(B), |b(x)| < 1 \}
\subseteq \text{closure of } \bigcup \{ \text{supp } x : x \in M(A)/M(B) \}
= N_A(B).
\]
Since this is true for any $b \in \mathcal{U}(B,A)$, we have $N_A(B) \supseteq W^*$. Thus $W^* = N_A(B)$ if $A$ is maximal in $B$.

It is trivial that if (iii) holds, $N_A(B) \subseteq N_A(b)$ for all $b \in \mathcal{U}(B,A)$.

We are done.

In Proposition 4.1 of [7] Izuchi constructed a family of Douglas algebras $B$ contained in $H'[q]$ with the property that $N(B) = N(q)$. By Proposition I, we have that this family is a family of maximal subalgebras of $H'[q]$.

Finally we close this paper with the following question that I have been unable to answer.

QUESTION 1. Recall that if $q$ is an interpolating Blaschke product, then $N(q) = N(B) \cup N_B(q)$ for any Douglas algebra with $B \subseteq H'[q]$. Does there exist a Douglas algebra $B_o \subseteq H'[q]$ with $N_{B_o}(q) = N(q)$?

REFERENCES
