SUMMABILITY METHODS BASED ON THE RIEMANN ZETA FUNCTION

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ABSTRACT. This paper is a study of summability methods that are based on the Riemann Zeta function. A limitation theorem is proved which gives a necessary condition for a sequence x to be zeta summable. A zeta summability matrix Z_t associated with a real sequence t is introduced; a necessary and sufficient condition on the sequence t such that Z_t maps l_1 to l_1 is established. Results comparing the strength of the zeta method to that of well-known summability methods are also investigated.

KEY WORDS AND PHRASES. Zeta summability method, zeta matrix method, I - I matrix, Cesaro method, Euler-Knopp method.

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1. INTRODUCTION.

Recall that the Riemann zeta function is given by $\zeta(s) = \sum_{k=1}^{\infty} (1/k^s)$ for $s > 1$ (Titchmarsh [1]).

A number sequence is said to be *zeta summable* to L (or ζ -summable to L) provided that

$$\lim_{s \rightarrow +\infty} \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{x_n}{n^s} = L.$$

The zeta method is a "sequence-to-function" summability method whose domain consists of those sequence x such that the Dirichlet's series $\sum_{k=1}^{\infty} (x_k/k^s)$ is convergent for $s > 1$.

In the second section it is shown that the zeta summability method is regular and totally regular (preserves finite and infinite limits). A limitation theorem is proved which gives a necessary condition for a sequence x to be zeta summable. In section 3 we introduce a zeta summability matrix Z_t associated with a real sequence t ; a necessary and sufficient condition on the sequence t such that Z_t maps l_1 into l_1 is established. The final section contains results comparing the strength of the zeta method to that of well-known summability methods. For example, the zeta method is stronger than the Cesàro method of order 1 but does not include the Cesàro method of order 2; the zeta method does not include and is not included in the Euler-Knopp method of order r for $0 < r < 1$.

2. BASIC THEOREMS

THEOREM 1. The ζ -summability method is totally regular.

Proof. First let x be a sequence satisfying $\lim_k x_k = L$, and suppose $\epsilon > 0$. Then choose N_1 so that $k > N_1$ implies $|x_k - L| < \epsilon/2$. Now for any positive integer k and $s > 1$ we see that $\sum_{k=1}^{N_1} (x_k/k^s) + (x_k - L)/(k^s)$ is

bounded by $\sum_{k=1}^{N_1} |x_k - L| = M$. Since $\sum_{k=1}^{\infty} 1/k = \infty$, we can choose $N_2 > N_1$ so that $\sum_{k=1}^{N_2} 1/k > (2M/\epsilon) + 1$. Now choose δ such that $0 < \delta < \log[1 + (1/N_2)]/\log N_2$. Then for each $k \leq N_2$, we have

$$k^\delta < k^{1/\epsilon \log[1 + (1/N_2)]/\log N_2} \leq 1 + (1/N_2);$$

and if $1 < s < 1 + \delta$

$$(1/k) - (1/k^s) < (k^\delta - 1)/k^s < k^\delta - 1 < 1/N_2.$$

Summing from $k = 1$ to N_2 , we obtain

$$\begin{aligned} \sum_{k=1}^{N_2} \left(\frac{1}{k^s} \right) &> \sum_{k=1}^{N_2} \left(\frac{1}{k} - \frac{1}{N_2} \right) \\ &> \left(\frac{2M}{\epsilon} \right) + 1 - 1 \\ &= \frac{2M}{\epsilon}. \end{aligned}$$

Thus for $1 < s < 1 + \delta$,

$$\begin{aligned} \zeta(s) &> \sum_{k=1}^{N_2} \frac{1}{k^s} \\ &> \frac{2M}{\epsilon}. \end{aligned}$$

and

$$\begin{aligned} \left| \frac{1}{\zeta(s)} \sum_{k=1}^{\infty} \frac{x_k}{k^s} - L \right| &\leq \frac{1}{\zeta(s)} \sum_{k=1}^{N_1} \frac{1}{k^s} |x_k - L| + \frac{1}{\zeta(s)} \sum_{k > N_1} \frac{1}{k^s} |x_k - L| \\ &< \frac{\epsilon}{2M} M + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence,

$$\lim_{s \rightarrow 1^+} \frac{1}{\zeta(s)} \sum_{k=1}^{\infty} \left(\frac{x_k}{k^s} \right) = L.$$

Now assume x is a real number sequence which diverges to ∞ . Then for each number $M > 0$ there exists a positive integer N such that $x_k > M + 1$ for all $k > N$. Suppose $s > 1$ and consider

$$\begin{aligned} \frac{1}{\zeta(s)} \sum_{k=1}^{\infty} \frac{x_k}{k^s} &> \frac{1}{\zeta(s)} \sum_{k=1}^N \frac{x_k}{k^s} + \frac{M+1}{\zeta(s)} \sum_{k > N} \frac{1}{k^s} \\ &= \frac{1}{\zeta(s)} \sum_{k=1}^N \left(\frac{x_k - M - 1}{k^s} \right) + (M+1). \end{aligned}$$

Since $\zeta(s) \rightarrow \infty$ as $s \rightarrow 1^+$, we see that if s is sufficiently close to 1 on the right, then

$$\left| \frac{1}{\zeta(s)} \sum_{k=1}^N \left(\frac{x_k - M - 1}{k^s} \right) \right| < 1 ;$$

this implies that

$$\begin{aligned} \frac{1}{\zeta(s)} \sum_{k=1}^{\infty} \frac{x_k}{k} &> \frac{1}{\zeta(s)} \sum_{k=1}^N \left(\frac{x_k - M - 1}{k^s} \right) + M + 1 \\ &> -1 + M + 1 \\ &= M . \end{aligned}$$

Since $M > 0$ was chosen arbitrarily, we conclude that

$$\lim_{s \rightarrow 1^+} \frac{1}{\zeta(s)} \sum_{k=1}^{\infty} \frac{x_k}{k^s} = \infty .$$

A previous definition of "zeta summability" was given in Diaconis [2]. In that paper the bounded sequence x is said to be zeta summable to L if

$$\lim_{s \rightarrow 1^+} (s-1) \sum_{i=1}^{\infty} \frac{x_i}{i^s} = L .$$

This is equivalent to the definition of the zeta method introduced in this paper. There equivalence is an immediate consequence of the fact that $\lim_{s \rightarrow 1^+} \zeta(s) (s-1) = 1$.

Recall that a Stoltz domain of angle α , where $0 < \alpha < \pi/2$, is a complex number set of the form

$$S(\alpha) = \{w : |\operatorname{Arg}(w-1)| < \alpha, \text{ and } |w| < 1\} .$$

(Powell et al [3]).

We shall use a variant of this concept, which we shall call a "reflected Stoltz domain of angle α ."

$$S^*(\alpha) = \{w : |\operatorname{Arg}(w-1)| < \alpha \text{ and } \operatorname{Re}(w) > 1\} .$$

This concept is now used to extend the zeta method to one using a complex-valued function limit, and we establish the regularity of this extension.

THEOREM 2. Let $S^*(\alpha)$ be a reflected Stoltz domain of angle α ; if the sequence x converges to L then

$$\lim_{w \rightarrow 1, w \in S^*(\alpha)} \frac{1}{\zeta(w)} \sum_{k=1}^{\infty} \left(\frac{x_k}{k^w} \right) = L .$$

The proof of Theorem 2 that we shall give needs the following preliminary result.

LEMMA 1. For $w = \sigma + it$, $w \in S^*(\alpha)$, and w sufficiently close to 1, we have

$$\frac{1}{|\zeta(w)|} \sum_{k=1}^{\infty} \left| \frac{1}{k^w} \right| \leq 2 \sec \alpha .$$

Proof. Since $\zeta(w)$ can be expanded in the form $(w-1)^{-1} + P(w-1)$, where $P(w-1)$ is a power series in $(w-1)$, (Hardy [4], p. 333), we have

$$\begin{aligned}
\frac{1}{|\zeta(s)|} \sum_{k=1}^{\infty} \frac{1}{k^w} &= \frac{|\zeta(\sigma)|}{|\zeta(w)|} \\
&= \frac{\left| \frac{1}{\sigma-1} + P(\sigma-1) \right|}{\left| \frac{1}{w-1} + P(w-1) \right|} \\
&\rightarrow \left| \frac{w-1}{\sigma-1} \right| \quad \text{as } w \rightarrow 1.
\end{aligned}$$

Since the limit value $|w-1|/|\sigma-1| \leq \sec \alpha$ for $w \in S^*(\alpha)$, this proves the assertion.

Now we prove Theorem 2.

Proof (of Theorem 2). Let $\varepsilon > 0$. Since x converges to L , we can choose $N_1 \ni |x_k - L| < (\varepsilon/4) \cos \alpha$ for $k \geq N_1$. Let $\sum_{k=1}^{N_1} |x_k - L| = M$. Since $\zeta(w) \rightarrow \infty$ as $w \rightarrow 1$, we have $1/\zeta(w) < \varepsilon/2M$ for w sufficiently close to 1.

Now for $w \in S^*(\alpha)$, we have

$$\begin{aligned}
\left| \frac{1}{\zeta(w)} \sum_{k=1}^{\infty} \frac{x_k}{k^w} - L \right| &\leq \frac{1}{|\zeta(w)|} \sum_{k=1}^{\infty} \frac{1}{|k^w|} \cdot |x_k - L| \\
&= \frac{1}{|\zeta(w)|} \left[\sum_{k=1}^{N_1} \frac{1}{|k^w|} |x_k - L| + \sum_{k > N_1} \frac{1}{|k^w|} |x_k - L| \right] \\
&< \frac{M}{|\zeta(w)|} + \frac{1}{|\zeta(w)|} \cdot \frac{\varepsilon}{4} \cos \alpha \sum_{k=1}^{\infty} \frac{1}{|k^w|} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} (\cos \alpha)^2 \sec \alpha \\
&= \varepsilon.
\end{aligned}$$

Next we prove a limitation theorem which asserts that the ζ -summability method cannot sum a sequence that diverges too rapidly.

THEOREM 3. If a complex number sequence x is ζ -summable, then for each $s > 1$, $x_n = o(n^s)$. Moreover, the term $o(n^s)$ is the best possible in the sense that the conclusion fails if n^s is replaced by any real sequence t_n such that t_n/n^s decreases to zero.

Proof. For x to be ζ -summable, x must be in the domain of the ζ -summability method. Therefore $\sum_{n=1}^{\infty} (x_n/n^s)$ converges for all $s > 1$, which implies that $\lim_n (x_n/n^s) = 0$. If n^s is replaced by t_n , where t_n/n^s decreased to 0, then we assert that it will not be true that $x_n = o(t_n)$ whenever x is ζ -summable. This is equivalent to showing that there is a sequence x such that x is ζ -summable and $x_n \neq o(t_n)$. Define the sequence x by $x_n = (-1)^{n+1} t_n$, so that

$$\sum_{n=1}^{\infty} \frac{x_n}{n^s} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t_n}{n^s}.$$

This is a convergent alternating series, and its (positive) sum is bounded by its first term t_1 .

Hence,

$$\lim_{s \rightarrow 1^+} \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{x_n}{n^s} = 0.$$

i.e., x is ζ -summable to 0. But $x_n \neq o(t_n)$ because for each n , $|x_n/t_n| = 1$.

3. ZETA SUMMABILITY MATRICES

Definition. Let t be a sequence of real numbers such that $t(n) > 1$ for every n and $\lim_n t(n) = 1$.

Then the *zeta matrix* $Z_t = [z_{nk}]$ associated with the sequence t is defined by

$$z_{nk} = \frac{1}{\zeta(t(n))k^{t(n)}} \quad \text{for } n, k = 1, 2, 3, \dots$$

In this section we make use of two well-known theorems in summability theory, which we shall subsequently cite by name only; they are Silverman-Toeplitz Theorem ([5] and [6]) and the Knopp-Lorentz Theorem [7]. It is an easy calculation to show that Z_t satisfies the conditions of the Silverman-Toeplitz Theorem for regularity. Moreover, Z_t is totally regular because all of its entries are positive real numbers ([3] p. 35). We summarize these observations in the following theorem.

THEOREM 4. The zeta matrix Z_t associated with the sequence t is totally regular.

The next result is a characterization of those sequences t for which Z_t is an I -matrix, i.e., Z_t maps I_1 into I_1 .

THEOREM 5. The matrix Z_t is an I -matrix if and only if $t-1$ is in I_1 .

Proof. Since each row sequence of the matrix Z_t is decreasing, the set of the sums of column sequences of the matrix Z_t is bounded by the sum of its first column entries. Therefore by the Knopp-Lorentz Theorem, it is enough to show that the first column sum is finite whenever $\sum_{n=1}^{\infty} (t(n) - 1)$ is convergent. This is a consequence of the inequality

$$\sum_{n=1}^{\infty} \frac{1}{\zeta(t(n))} \leq \sum_{n=1}^{\infty} (t(n) - 1),$$

which follows immediately from the fact that for $s > 1$,

$$\begin{aligned} \frac{s-1}{s} &\leq \frac{1}{\zeta(s)} \\ &\leq s-1 \end{aligned} \quad (*)$$

Hence Z_t is an I -matrix.

Conversely, assume Z_t maps I_1 to I_1 . Since $t(n) > 1$ and $\lim_n t(n) = 1$ for every n , we can choose a positive integer N such that $0 < t(n) - 1 < 1$ for $n \geq N$. Suppose $t-1$ is not in I_1 ; then

$$\begin{aligned} \sum_{n=N}^{\infty} \left(\frac{1}{t(n)} \right) &= \sum_{n=N}^{\infty} \left(\frac{t(n)-1}{t(n)} \right) \\ &> \sum_{n=N}^{\infty} \left(\frac{t(n)-1}{2} \right) \\ &= \infty. \end{aligned}$$

Now $\sum_{n=1}^{\infty} (1/\zeta(t(n)))$ diverges to infinity because of the inequality $1/\zeta(t(n)) \geq (1 - 1/t(n))$ as in (*).

Therefore, by the Knopp-Lorentz Theorem, Z_1 is not an I -matrix. This completes the proof of the theorem.

4. INCLUSION THEOREMS.

In this section we compare the strength of the zeta method and the zeta matrix methods to several well-known summability methods. Throughout this section C_α denotes the Cesaro summability matrix of order α and E_r the Euler-Knopp summability matrix of order r .

LEMMA 2. If x is a sequence that is C_1 -summable, then x is in the domain of the ζ -summability method, and hence, x is in the domain of every Z_1 method.

Proof. Assume that x is C_1 -summable to L : $\lim_n (x_1 + \dots + x_n)/n = L$. To get the conclusion it is enough to show that the abscissa of convergence σ_o of the Dirichlet series $\sum_{n=1}^{\infty} x_n/n^s$ is less than or equal to 1, where σ_o is given by

$$\sigma_o = \limsup_{n \rightarrow \infty} \frac{\log \left| \sum_{k=1}^n x_k \right|}{\log n}.$$

(Hardy et al [8] or Titchmarsh [9]). Since x is c_1 -summable to L , there exists a positive integer N such that if $n \geq N$, then

$$\frac{\left| \sum_{k=1}^n x_k \right|}{n} \leq |L| + 1.$$

This implies that $\left| \sum_{k=1}^n x_k \right| \leq n(|L| + 1)$, so

$$\log \left| \sum_{k=1}^n x_k \right| \leq \log [n(|L| + 1)].$$

Therefore

$$\begin{aligned} \sigma_o &= \limsup_{n \rightarrow \infty} \frac{\log \left| \sum_{k=1}^n x_k \right|}{\log n} \\ &\leq \limsup_{n \rightarrow \infty} \left(\frac{\log n (|L| + 1)}{\log n} \right) \\ &= 1. \end{aligned}$$

THEOREM 6. The Z_1 method includes the C_1 method.

Proof. This inclusion is equivalent to the regularity of the matrix $Z_1 C_1^{-1}$, which can be verified by direct calculation using the Silverman-Toeplitz Theorem.

The following example shows that the C_1 method does not include the Z_1 method.

EXAMPLE.

Let $x = \{(-1)^k k\}$; then

$$\begin{aligned} (Z_t x)_n &= \sum_{k=1}^{\infty} \frac{(-1)^k k}{s(t(n))k^{t(n)}} \\ &= \frac{1}{s(t(n))} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{t(n)-1}} \end{aligned}$$

Since

$$\begin{aligned} -1 &\leq \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{t(n)-1}} \\ &\leq 0 \quad \text{for } t(n) > 1 \end{aligned}$$

and $\lim_n \zeta(t(n)) = \infty$, it is easy to see that $\lim_n (Z_t x)_n = 0$. On the other hand, we have

$$\begin{aligned} (C_1 x)_n &= \frac{1}{n} \sum_{k=1}^n (-1)^k k \\ &= \begin{cases} \frac{1}{2}, & \text{if } n \text{ is even} \\ -\frac{1}{2}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Thus $\lim_n (C_1 x)_n$ does not exist, so x is not C_1 -summable.

By a "continuous parameter sequence-to-function transformation", we mean a summability method F that is determined as follows by a function sequence $\{f_k(z)\}_{k=1}^{\infty}$: for a given sequence x form the function

$$F_x(z) = \sum_{k=1}^{\infty} f_k(z)x_k ; \quad (**)$$

if $\lim_{z \rightarrow a} F_x(z) = L$, then we say that " x is F -summable to L ". For a given function sequence $\{f_k(z)\}_{k=1}^{\infty}$ and a given number sequence t , we can also form an associated matrix F_t , which is given by

$$F_t[n, k] = f_k(t(n)) .$$

The next lemma, which will be used to compare the C_1 method and the ζ method, is a comparison of the method F and the associated matrix method F_t .

LEMMA 3. Let F be a continuous parameter sequence-to-function transformation as in (**) and define the sequence sets

$$S_F = \{x : \lim_{z \rightarrow a} F_x(z) \text{ exists} \} ,$$

$$S_{F_t} = \{x : F_t x \text{ is convergent} \} ,$$

and

$$T = \{t : \lim_n t(n) = a\} ;$$

then

$$S_F = \bigcap_{t \in T} S_{F_t}.$$

Proof. We show that each of S_F and $\bigcap_{t \in T} S_{F_t}$ contains the other. Since F_t includes F for $t \in T$, we have

$$S_F \subseteq \bigcap_{t \in T} S_{F_t}.$$

To prove the reverse inclusion, we consider a sequence x which is not in S_F . It follows that $\lim_{z \rightarrow a} F_X(z)$ does not exist. By the sequential criterion for function limits (Almsted [10], p. 73), there is a sequence t_n in T such that $\lim_n (F_{t_n} x)_n$ does not exist. This implies that x is not in the set $S_{F_{t_n}}$. Hence x is not in the set $\bigcap_{t \in T} S_{F_t}$.

THEOREM 7. The ζ -summability method is stronger than the C_1 method.

Proof. By Lemma 3, we have $S_\zeta = \bigcap_{t \in T} S_{Z_t}$. Since the Z_t method includes the C_1 method for all t in T , we have $S_{C_1} \subseteq \bigcap_{t \in T} S_{Z_t} = S_\zeta$. Now if x is a sequence that is C_1 -summable to L , then x is Z_t summable to L for all t in T . Therefore the sequential criterion for function limits ensures that x is ζ -summable to L . Hence, the ζ method includes the C_1 method. It is easy to see that the C_1 method does not include the ζ method because C_1 method does not include the Z_t method.

As a consequence of Theorem 6, we can infer that Z_t includes any method that is included by C_1 . For example, Z_t includes the divisor method D_r for $r > 0$. (Fridy [11]).

Let H_2 denote the Holder method of order 2. By arguing as in the proof of Theorem 6, we can prove

THEOREM 8. If the sequence x is H_2 -summable to L and x is in the domain of the Z_t method, then x is Z_t summable to L .

COROLLARY. If the sequence x is H_2 -summable to L and x is in the domain of the ζ -summability method, then x is ζ -summable to L .

The conclusion of the preceding Corollary does not hold if x is not in the domain of the ζ method. This is shown by the following example.

EXAMPLE. Let x be the sequence defined by

$$x_n = \begin{cases} (-1)^k k^{\frac{3}{2}}, & \text{if } n=2k, k=1,2,\dots \\ (-1)^{k+1} k^{\frac{3}{2}}, & \text{if } n=2k-1, k=1,2,\dots \end{cases}$$

If $x \leq 3/2$, then the series $\sum_{n=1}^{\infty} (x_n/n^s)$ is divergent because its n^{th} term does not approach 0. Therefore x is not in the domain of the ζ method, and hence, x is not ζ -summable. Now we show that x is H_2 -summable to zero. Since $(C_1 x)_{2k-1} = (-1)^{k+1} k^{3/2}/(2k-1)$ and $(C_1 x)_{2k} = 0$, we see that the (odd) partial sums alternate in sign after $k = 3$; thus the partial sum is not greater than the last term, which is $O(k^{1/2})$.

Therefore, upon dividing by 2^{k-1} to form $C_1(C_1x)_{2^{k-1}}$, we have

$$\begin{aligned}(H_2x)_{2^{k-1}} &= \left\{ \frac{1}{2^{k-1}} \right\} O \left\{ k^{\frac{1}{2}} \right\} \\ &= O \left\{ k^{-\frac{1}{2}} \right\} \\ &= o(1),\end{aligned}$$

which proves that x is H_2 -summable to zero.

Since the Holder method of order 2 is equivalent to the Cesaro method of order 2 (Hardy [4], p. 103), we can immediately get the following theorem.

THEOREM 9. If x is a sequence which is C_2 -summable to L and x is in the domain of the summability method, then x is ζ -summable to L .

It is well known that for each number r satisfying $0 < r < 1$ and any nonzero real number α , E_r . By using these facts, we have the following result.

THEOREM 10. The ζ method is not included in E_r for $0 < r < 1$.

The following example shows that the ζ method does not include E_r for $0 < r < 1$.

EXAMPLE. Given r between 0 and 1 choose $\epsilon > 0$ satisfying $r < 2 / (2 + \epsilon)$. Next define $x_k = (-1-\epsilon)^k$.

Then

$$\begin{aligned}(E_r x)_n &= \sum_{k=0}^n \binom{n}{k} r^k (1-r)^{n-k} (-1-\epsilon)^k \\ &= [(-1-\epsilon)r + (1-r)]^n \\ &= [(-2-\epsilon)r + 1]^n.\end{aligned}$$

Since $0 < r < 2 / (2 + \epsilon)$, we have $-1 < (-2-\epsilon)r + 1 < 1$. This implies that

$$\begin{aligned}\lim_n (E_r x)_n &= \lim_n [(-2-\epsilon)r + 1]^n \\ &= 0,\end{aligned}$$

i.e., x is E_r -summable to 0. But x is not in the domain of the ζ method because the series

$$\sum_{k=1}^{\infty} \frac{(-1-\epsilon)^k}{k^s}$$

is not convergent for any s , whence x is not in the domain of the ζ method.

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