WEAKLY RECURRENT AND ALMOST PERIODIC POINTS IN βG .

MOSTAFA NASSAR

Department of Mathematics
Acadia University
Wolfville, N.S. BOP 1X0
Canada

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ABSTRACT. Let G be a group. We will study the relationship between A^G and WR^G . In Nassar [1] it has been shown that if G is a nontorsion group, then $A^G \subsetneq R^G$. In this paper we will show that if G contains a subset A such that KA is not left thick for each finite set K and A is not a finite union of thin sets, then $A^G \subsetneq WR^G$.

KEY WORDS AND PHRASES. *Topological group, recurrent point, Stone-Ćech compactification, almost periodic point.*

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I. INTRODUCTION

This paper is a sequel to Nassar [1] and we shall freely use the notations as presented in [1].

If S is a discrete semigroup, then for $s \in S$ the left translation ℓ_s is a linear map from $M(S)$ the Banach space of all bounded real valued functions on S into $M(S)$ given by

$$(\ell_s f)(s') = f(ss') \text{ for all } s' \in S.$$

Denote by $LIM(S)$ the set of left invariant means on $M(S)$. If $\varphi \in LIM(S)$ then φ can be considered as a member of $C(\beta S)^*$. By the Riesz representation theorem then there exists a bounded regular

Borel measure $\mu_\varnothing$ on βS such that

$$\varnothing(f) = \int_{\beta S} \tilde{f} \, d\mu_\varnothing, \quad f \in M(S).$$

DEFINITION. For a subset H of a left amenable semigroup S , define the upper and lower densities as follows:

$$\overline{d}(H) = \sup\{\varphi(I_H) : \varnothing \in \text{LIM}(S)\} = \sup\{\mu_\varnothing(\tilde{H}) : \varnothing \in \text{LIM}(S)\}$$

$$\underline{d}(H) = \inf\{\varphi(I_H) : \varnothing \in \text{LIM}(S)\} = \inf\{\mu_\varnothing(\tilde{H}) : \varnothing \in \text{LIM}(S)\}$$

$$\text{put } K^S = \text{cl}[U\{\text{supp}\mu_\varnothing : \varnothing \in \text{LIM}(S)\}]$$

Let G be a group. Then G acts as group of homeomorphisms from βG into βG . Using the terminology of topological dynamics, $(G, \beta G)$ is a flow. In fact, for a fixed G , the flow $(G, \beta G)$ is universal. Therefore, it is interesting to study almost periodic points and several types of points in βG as given in the following definitions.

DEFINITION. (1) $w \in \beta G$ is recurrent if whenever U is a neighbourhood of w then the set $\{x \in G : xw \in U\}$ is infinite. Let R^G be the set of all recurrent points in βG .

(2) $w \in \beta G$ is discrete if its orbit $O(w)$ is discrete with respect to the subspace topology of βG . Let D^G be the set of all such points in βG .

(3) $w \in \beta G$ is strongly discrete if there exists a neighbourhood U of w such that $xU \cap yU = \varnothing$ if $x, y \in G$, $x \neq y$. The set of all strongly discrete points will be denoted by SD^G . Let $WR^G = \beta G - SD^G$, the set of all weak recurrent points of βG , and A^G be the set of all almost periodic points of βG .

If G is finite group, the $A^G = D^G = SD^G = K^G = G = \beta G$ and $R^G = WR^G = \varnothing$. Therefore we are only interested in infinite groups. For the remainder of this paper, G stands for an infinite group.

DEFINITION. (1). A subset A of a group G is called left thick if, for every finite subset K of G there exists $g \in G$ such that $Kg \subset A$. Mitchell [2] proved that if S is left amenable semigroup and $A \subseteq S$,

then $\bar{d}(A) = 1$ if and only if A is left thick.

(2) THIN SETS AND ALMOST PERIODIC POINTS.

DEFINITION: A subset A of G is said to be thin if $g_1 A \cap g_2 A$ is a finite subset of G for each distinct pair of elements $g_1, g_2 \in G$.

LEMMA 2.1. For an arbitrary subset B of a group G there exists a subset $A \subseteq B$ with $|A| = |B|$ such that A is thin.

The proof of this lemma can be found in Chou [3;p.202].

EXAMPLES 2.2. (1) The sets $\{n! : n \in \mathbb{N}\}, \{2^n : n \in \mathbb{N}\}$ are infinite thin sets in $\mathbb{Z}$.

(2) Consider the infinite abelian periodic group $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \dots$. Let A be an infinite subset of G such that for each n , the set of all $x = (x_k) \in A$ with $x_k \neq 0$ for some $k \leq n$ is finite. Now for each pair of distinct elements $g_1, g_2 \in G$, $|(g_1 + A) \cap (g_2 + A)|$ needs to be calculated. But $g_1 \neq g_2$ if and only if $g = g_1 + g_2 \neq e$, so it is enough to calculate $|A \cap (g + A)|$ if $g \neq e$. For any $g = (x_1, x_2, x_3, \dots, x_k, \dots)$ where the first non-zero component appears in the k -th position, denote by A_1 the set of all $x = (x_k) \in A$ with $x_k \neq 0$. By assumption, A_1 is a finite subset of A . Put $A = A_1 \cup A_2$ ($\cup$ means the disjoint union.)

$$\begin{aligned} A \cap (g + A) &= (A_1 \cup A_2) \cap (g + A) \\ &= (A_1 \cap (g + A)) \cup (A_2 \cap (g + A)) \\ &= (A_1 \cap (g + A)) \cup \{A_2 \cap (g + (A_1 \cup A_2))\} \\ &= (A_1 \cap (g + A)) \cup (A_2 \cap (g + A)) \cup (A_2 \cap (g + A_2)). \end{aligned}$$

On the right hand side, the cardinalities of the first and the second term are finite and the last term is empty. Then $|A \cap (g + A)|$ is finite, which implies A is a thin subset of G .

PROPOSITION 2.3. SD^G is dense in βG .

PROOF: To show that SD^G is dense in βG it suffices to show that $SD^G - G$ is dense in $\beta G - G$. By lemma 2.1, for each non-empty closed-open subset $\hat{B}$ of $\hat{G} = \beta G - G$ there exists a non-empty subset $\hat{A} \subseteq \hat{B}$ such that A is a thin subset of G . But by theorem 2.6 part 2 of Nassar [1] $SD^G - G = \{\hat{A} : A \text{ is a thin subset of } G\}$ and hence from the

fact that the closed-open sets form a topological basis of $\beta G - G$ we conclude that $SD^G - G$ is dense in $\beta G - G$. So SD^G is dense in βG .

LEMMA 2.4. For $w \in \hat{G}$, $w \in K^G$ if and only if whenever $w \in \hat{A}$ we always have $\bar{d}(A) > 0$.

PROOF. Let $w \in K^G$ and suppose $w \in \hat{A}$. Since $\hat{A}$ is open and $K^G = \text{cl}[U\{\text{supp } \mu_\varphi : \varphi \in \text{LIM}(G)\}]$, so there exists $\varphi \in \text{LIM}(G)$ such that $\hat{A} \cap \text{supp } \mu_\varphi \neq \emptyset$ which implies $\mu_\varphi(\hat{A}) > 0$ implies $\bar{d}(A) > 0$. Conversely if for each closed-open neighbourhood $\hat{A}$ of w , $\bar{d}(A) > 0$ then there exists $\varphi \in \text{LIM}(G)$ such that $\mu_\varphi(\hat{A}) > 0$ implies $\hat{A} \cap \text{supp } \mu_\varphi \neq \emptyset$ thus $w \in K^G$.

PROPOSITION 2.5 (i) $K^G \cap SD^G = \emptyset$

(ii) K^G is nowhere dense in βG .

PROOF: (i) By lemma 2.4, if $w \in K^G$ and if $\bar{A}$ is a closed-open neighbourhood of w then $\bar{d}(A) > 0$. Theorem 2.6 of [1], implies that A is not thin and hence $w \notin SD^G$.

(ii) By proposition 2.3 SD^G is dense in βG and by part (i) $K^G \cap SD^G = \emptyset$. So K^G is nowhere dense in βG .

PROPOSITION 2.6. If A is not a finite union of thin sets, then there exists $w \in \bar{A}$ such that $w \notin SD^G$.

PROOF: Suppose $\bar{A} \subset SD^G$. For each $w \in \bar{A}$, let $\bar{B}_w$ be a closed-open neighbourhood of w so that B_w is a thin subset. Then $\{\bar{B}_w : w \in \bar{A}\}$ is an open cover of the compact subset $\bar{A}$. This implies the existence of a finite subcover of $\bar{A}$ say, $\bar{B}_{w_1}, \bar{B}_{w_2}, \dots, \bar{B}_{w_n}$. Then $B_{w_1} \cup B_{w_2} \cup \dots \cup B_{w_n} = A$. Hence if A is not a finite union of thin sets, then there exists $w \in \bar{A}$ such that $w \notin SD^G$, i.e. $w \in WR^G$.

LEMMA 2.7. Let $A \subset G$. Then the following are equivalent.

(i) $\bar{A} \cap A^S \neq \emptyset$

(ii) $\bar{A} \cap \bar{A}^S \neq \emptyset$

(iii) There exists $\{s_1, s_2, \dots, s_{r-1}\} \subset G$ such that

$$\bar{d}(AU_{s_1}AU_{s_2}AU \dots U_{s_{r-1}}A) = 1$$

PROOF. (i) implies (ii) is clear

(ii) implies (i): If $w \in \bar{A} \cap A^S$, $\bar{A}$ is open, then there exists a neighbourhood V of w which meets A^S . Hence there exists $w_0 \in \bar{A} \cap A^S$.

(i) implies (iii): $w \in \bar{A} \cap A^S$ implies $B = \{s \in G : sw \in \bar{A}\}$ is relatively dense, so there exists $s_1, s_2, \dots, s_n$ in G such that $s_1B \cup s_2B \cup \dots \cup s_nB = G$. Thus for $s \in G$ there is $s_i \in G$ such that $s_i^{-1}s \in B$ and hence $s_i^{-1}sw \in \bar{A}$ and $sw \in s_iA$. Therefore $\bar{O}(w) \subset \bigcup_{i=1}^n s_iA$. So there exists a $\emptyset \in \text{LIM}(G)$ such that $\text{supp } \mu_\emptyset \subset \bar{O}(w)$, and hence $\mu_\emptyset(\bigcup_{i=1}^n s_i\bar{A}) = 1$.

(iii) implies (i): If $\bar{d}(AU_{s_1}AU \dots U_{s_{r-1}}A) = 1$, then $\bar{A}U_{s_1}\bar{A}U \dots U_{s_{r-1}}\bar{A}$ must contain G -minimal set which then will meet $\bar{A}$ and (i) follows, which completes the proof.

PROPOSITION 2.8 If G contains a subset A such that $K.A$ is not left thick for each finite set K and A is not a finite union of thin sets,

then $\bar{A}^G \subsetneq \text{WR}^G$.

PROOF: Since $K.A$ is not left thick for each finite subset K of G , by lemma 2.7, $\bar{A} \cap A^G = \emptyset$. But proposition 2.6 shows that there exists a $w \in \bar{A} \cap \text{WR}^G$. Thus $\bar{A}^G \subsetneq \text{WR}^G$. In other words there is a strongly recurrent point in βG which is not almost periodic point.

PROPOSITION 2.9. If an amenable group G contains an infinite subgroup H of infinite index, then $\bar{d}(H) = 0$ and there exists $w \in \beta H$ such that $w \in R^G$.

PROOF. Since $[G:H] = \infty$, the number of disjoint cosets $\{tH\}$ is infinite. For any $\emptyset \in \text{LIM}(G)$, since $\emptyset(I_H) = \emptyset(I_{t_1H})$ if $t_1H, t_2H, \dots, t_nH$

are distinct cosets then

$$\begin{aligned} \emptyset (t_1 H \cup t_2 H \dots \cup t_n H) &= \varphi(t_1 H) + \varphi(t_2 H) + \dots + \varphi(t_n H) \\ &= n\varphi(t_1 H) < 1 \end{aligned}$$

since n is arbitrary, $\varphi(t_1 H) = 0$ so $\bar{d}(H) = 0$. Note that if $w \in R^H$ and

if U is a neighbourhood of w in βG then $U \cap \bar{H}$ is also a neighbourhood of w and hence $X = \{h \in H : hw \in U \cap \bar{H}\}$ is infinite. Therefore $\{g \in G : gw \in U\} (\supset X)$ is infinite. Thus $w \in R^G$. Since $\emptyset \neq A^H \subset R^H$, $R^G \cap \bar{H} \neq \emptyset$.

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