SMOOTH STRUCTURES ON SPHERE BUNDLES OVER SPHERES

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ABSTRACT. In [1] R. De Sapio gave a classification of smooth structures of a p-sphere bundle over a q-sphere with one cross-section and \( p < q \). In [2] J. Munkres also gave a classification up to concordance of differential structures in the case where the bundle has at least two cross-sections. In [3] R. Schultz gave a classification in the case \( p \geq q \). Here we will give a classification of the p-sphere bundle over a q-sphere without any cross-section and \( p < q \).

KEY WORDS AND PHRASES. Smooth structures, differential classification, internal groups.


1. INTRODUCTION

Let \( E \) represent p-sphere bundle over a q-sphere with \( \beta \in \pi_{q-1}SO(p+1) \) the characteristic class of the corresponding p+1-disc bundle over the q-sphere. In [4] R. De Sapio gave a complete classification of the special case where \( \beta = 0 \). In [5] and [6] Kawakubo and Schultz respectively also gave a classification of \( E \) for this special case. This author in [7] gave a generalization of this special case to product of three ordinary spheres. In [1] a classification of \( E \) was given for \( p < q - 1 \) and where \( E \) has a cross-section and \( \beta \neq 0 \). In [3] Schultz gave a classification of \( E \) for \( p \geq q \) and \( E \) is without cross-section. We shall here remove the fact that \( E \) has a cross-section so that not every element of \( \pi_{q-1}SO(p+1) \) can be pulled back to the element \( \pi_{q-1}SO(p) \) in the homomorphism \( S_* : \pi_{q-1}SO(p+1) \to \pi_{q-1}SO(p) \) induced by the inclusion \( s : SO(p) \to SO(p+1) \). \( S^n \) denotes the unit n-sphere with the usual differential structure in the Euclidean...
(n+1)-space $\mathbb{R}^{n+1} \times \mathbb{Z}^n$ denotes an homotopy n-sphere and $\pi^n$ denotes the group of homotopy n-spheres. $H(p,k)$ denotes the subset of $\pi^p$ which consists of those homotopy p-sphere $\Sigma^p$ such that $\Sigma^p \times S^k$ is diffeomorphic to $S^p \times S^k$. By [4, Lemma 4], $H(p,k)$ is a subgroup of $\pi^p$ and it is not always zero and in fact in [7] we showed that if $k \geq p-3$, $H(p,k) = \pi^p$. We shall adopt the notation $E(\mathbb{Z}^q)$ to represent the total space of a p-sphere bundle over a homotopy q-sphere $\mathbb{Z}^q$. We will then prove the following:

**Theorem.** If M is a smooth, n-manifold homeomorphic to a p-sphere bundle over a q-sphere with total space E where $n = p+q \geq 6$ and $p < q$ then there exists homotopy spheres $\mathbb{Z}^q$ and $\mathbb{Z}^n$ such that M is diffeomorphic to $E(\mathbb{Z}^q) \# \mathbb{Z}^n$. We shall define a pairing $G : \pi^p SO(q) \times \pi_{q-1} SO(p+1) \rightarrow \pi^{p+q}$

and show that if $\beta \in \pi_{q-1} SO(p+1)$ is the characteristic class of a p-sphere bundle over an homotopy q-sphere $\mathbb{Z}^q$, then $G(\pi^p SO(q), \beta)$ equals the inertial group of $E(\mathbb{Z}^q)$. The above theorem together with the latter will give us the following.

**Theorem.** Let E be the total space of a p-sphere bundle over a q-sphere then the diffeomorphism classes of $(p+q)$-manifolds that are homeomorphic to E are in one-to-one correspondence with the group $\left( \frac{\pi^q}{H(p,q)} \times \frac{\pi^n}{\text{Image } \beta} \right)$ where $n = p+q \geq 6$ and $p < q$.

2. **Classification Theorem**

In this section, we will prove the classification theorem for any manifold $M^n$ homeomorphic to E. We will apply the obstruction theory to smoothing of manifolds developed by Munkres in [8]. Since $p+q \geq 6$ and $2 \leq p < q$ then E is simply-connected and the homology of E has no 2-torsion, hence the "Hauptvermutung" of D. Sullivan [9] applies and this means that piecewise linear homeomorphism can be replaced by homeomorphism, we shall not distinguish the two.

**Definition.** Let M and N be smooth closed n-manifolds and L a closed subset of M of dimension less than n. Let $f : M \rightarrow N$ be a homeomorphism such that for each simplex $\gamma$ of L, $\bar{\gamma}$ and $f(\bar{\gamma})$ are contained in coordinate systems under which they are flat. $f$ is said to be a diffeomorphism modulo L if $f|(M-L)$ is a diffeomorphism and each simplex $\gamma$ of L has a neighborhood V such that $f$ is smooth on $V-L$ near $\gamma$. By [8, Theorem 2.8], if M and N are homeomorphic then there is a diffeomorphism modulo (n-1)-skeleton of M. If $f : M \rightarrow N$ is a diffeomorphism modulo m-skeleton $m < n$ then the obstruction to deforming
f to a diffeomorphism modulo (m-1)-skeleton g : M → N is an element λ(f) ∈ H_m(M, r^{n-m}) where r^{n-m} is a group of diffeomorphism of S^{n-m-1} modulo those that extend to diffeomorphisms of D^{n-m}. g is called the smoothing of f. If λ(f) = 0 then by [8, §4] smoothing g exist.

**THEOREM 2.1.** If M is a smooth n-manifold homeomorphic to E where E denotes the total space of a p-sphere bundle over a q-sphere, 2 ≤ p < q and n = p + q then there exist homotopy spheres S^q and S^n such that M is diffeomorphic to E(S^q) # S^n where E(S^q) denotes the total space of a p-sphere bundle over the homotopy q-sphere S^q.

**PROOF.** E is the total space of a p-sphere bundle over a q-sphere with characteristic class [b] ∈ π_q SO(p+1) then E = D^q × S^p U D^q × S^p where f_b : S^{p-1} × S^p + S^{q-1} × S^p is a diffeomorphism defined by f_b(x,y) = (x,b(x),y), (x,y) ∈ S^{p-1} × S^p

\[
H_i(E) = \begin{cases} \mathbb{Z} & \text{for } i = 0, p, q, p+q \\ 0 & \text{elsewhere} \end{cases}
\]

Since M_n is homeomorphic to E where n = p+q ≥ 6 2 ≤ p < q, then M_n is simply connected and since H_3(M,Z) has no 2-torsion, then "Hauptvermutung" of D. Sullivan [9] implies that there is a piecewise linear homeomorphism h : M_n → E which by [8, §5] is a diffeomorphism modulo (n-1)-skeleton. Since H_i(M,Z) = 0 for n-p+1 ≤ i ≤ n-1 then we can assume that h is a diffeomorphism modulo n-p = q skeleton. The obstruction to a diffeomorphism modulo q-1 skeleton is λ(h) ∈ H_q(M, r^n) = r^P. If [φ] = λ(h) ∈ r^P where φ : S^{p-1} + S^{q-1} is a diffeomorphism that represents λ(h) and let ε^P denote the homotopy p-sphere where ε^P = D^p_1 U D^p_2. We define a map

j : S^p + ε^P where S^p = D^p_1 U D^p_2

such that

\[
j(x) = \begin{cases} x & \text{if } x ∈ D^p_1 \\ \varphi^{-1} \left( \frac{x}{|x|} \right) & \text{if } x ∈ D^p_2. \end{cases}
\]

So j is an homeomorphism which is identity on D^p_1 and the radial extension of φ^{-1} on D^p_2 and so the first obstruction λ(j) to deforming j to a diffeomorphism is [φ^{-1}] = -λ(h).

We then define id × j : D^q × S^p + D^q × ε^P where id is the identity, then id × j is a homeomorphism and it follows from [8, Def. 3.4] that the first obstruction λ(id × j) to
deforming \( \text{id} x j \) to a diffeomorphism is also \(-\lambda(h)\). We can form a manifold \( E' \) by identifying two copies of \( D^q \times S^p \) along their common boundaries \( S^{q-1} \times S^p \) by the diffeomorphism \( f_b : S^{q-1} \times S^p \to S^{q-1} \times S^p \) where \( f_b(x,y) = (x,b(x \cdot y)) \) and \([b] \in \pi_{q-1}SO(p+1)\). So \( E' = D^q \times S^p \cup f_b D^q \times S^p \). We define a map

\[
g : E = (D^q \times S^p)_1 \cup f_b (D^q \times S^p)_2 \cup (D^q \times S^p)_1 \cup f_b (D^q \times S^p)_2 = E' \text{ by } g(x,y) = \text{id} \times j(x,y)
\]
on both \((D^q \times S^p)_1\), and \((D^q \times S^p)_2\), the map looks like

\[
g = \text{id} \times j
\]

\(g\) is an homeomorphism and the first obstruction to a diffeomorphism is \(\lambda(\text{id} x j) = -\lambda(h)\). It follows that the obstructions to smoothing the composition \(g \cdot h : M \to E'\) is \(\lambda(g \cdot h) = \lambda(g) + \lambda(h) = -\lambda(h) + \lambda(h) = 0\). It follows that \(g \cdot h : M \to E'\) is a diffeomorphism modulo \((q-1)\)-skeleton. However in [7, Remark 1] we showed that \(D^q \times S^p\) is diffeomorphic to \(D^q \times S^p\) if \(p = q + 2\) and so by our hypothesis \(p < q\) then it follows that \(D^q \times S^p\) is diffeomorphic to \(D^q \times S^p\). This implies that \(E\) and \(E'\) are diffeomorphic hence \(g' : M \to E\) is a diffeomorphism modulo \((q-1)\)-skeleton. Since \(H_i(M, \mathbb{Z}) = 0\) for \(p + 1 \leq i \leq q - 1\), there is no more obstruction to deforming \(g'\) to a diffeomorphism until we get to \((p-1)\) skeleton. We can then assume that \(g'\) is a diffeomorphism modulo \(p\)-skeleton. The first obstruction to deforming \(g'\) to a diffeomorphism modulo \((p-1)\)-skeleton is \(\lambda(g';) \in H_p(M, r^q) = r^p\). Let \([\phi] = \lambda(g';) \in \pi^q\) where \(\phi : S^{q-1} \to S^{q-1}\) is a diffeomorphism which represents \(\lambda(g';) \in \pi^q\). We define \((\phi \cdot \text{id}) : S^{q-1} \times S^p \to S^{q-1} \times S^p\) where \((\phi \cdot \text{id})(x,y) = (\phi(x), y)\) and if \(b = [b] \in \pi_{q-1}SO(p+1)\) we also define \(f_b : S^{q-1} \times S^p \to S^{q-1} \times S^p\) where \(f_b(x,y) = (x, b(x \cdot y))\). We then have two orientation preserving diffeomorphisms of \(S^{q-1} \times S^p\) unto itself which we can compose to get \((\phi \cdot \text{id}) \cdot f_b : S^{q-1} \times S^p + S^{q-1} \times S^p\) where \((\phi \cdot \text{id}) \cdot f_b(x,y) = (\phi(x), b(x \cdot y))\). We then construct a manifold by attaching two copies of \(D^q \times S^p\) along their common boundary \(S^{q-1} \times S^p\) using the diffeomorphism \((\phi \cdot \text{id}) \cdot f_b\) to have \(D^q \times S^p \cup (\phi \cdot \text{id}) \cdot f_b\). Notice that this manifold is a \(p\)-sphere bundle over a homotopy \(q\)-sphere \(\varepsilon^q = D^q_1 \cup b D^q_2\) whose characteristic map is
We define a map $h : D^q \times S^p \cup D^q_2 \times S^p + D^q_1 \times S^p \cup D^q_2 \times S^p$ by

$$h(x, y) = \begin{cases} (x, y) & \text{if} \ (x, y) \in D^q_1 \times S^p \\ (x, \phi^{-1}(\frac{x}{|x|}), y) & \text{if} \ (x, y) \in D^q_2 \times S^p \end{cases}$$

Hence $h$ is identity on $D^q_1 \times S^p$ and a radial extension of $\phi^{-1}$ on $D^q_2$. It then follows that $h$ is a homeomorphism with the first obstruction to a diffeomorphism being $[-\lambda(g')]$. Then by [8, 3.8] the first obstruction to deforming the composition $g' \circ h = g : M + D^q_1 \times S^p \cup D^q_2 \times S^p$ into a diffeomorphism is $\lambda(g) = \lambda(g', h) = \lambda(g') + \lambda(h) = -\lambda(h) + \lambda(h) = 0$ and hence $g$ is a diffeomorphism modulo $(p-1)$-skeleton. Since $H_i(M, Z) = 0$ for $0 < i < p$ then we can assume that $g$ is a diffeomorphism modulo one point.

3. INERTIAL GROUPS

Since by Theorem 2.1, every manifold homeomorphic to $E$ is diffeomorphic to $E(\Sigma^q) \# \xi^n$ for some homotopy spheres $\Sigma^q, \xi^n$, classification of such manifolds reduces to classification of manifolds of the form $E(\Sigma^q) \# \xi^n$. To complete this classification, we then need to investigate what happens when we vary the homotopy spheres and in particular we need to investigate the Inertial group of $E(\Sigma^q)$. We will investigate these in this section.

**Lemma 3.1.** Let $\Sigma^q_i$ and $\Sigma^q_2$ be homotopy q-spheres such that $\Sigma^q_i = D^q_i \cup \Sigma^q_2$ for $i = 1, 2$, then $E(\Sigma^q_i)$ is diffeomorphic to $E(\Sigma^q_2)$ if and only if $\Sigma^q_1 \pm \Sigma^q_2 \in \pi_{q-1}SO(p+1)$.

**Proof.** Suppose $E(\Sigma^q_1)$ is diffeomorphic to $E(\Sigma^q_2)$. This means that $D^q_1 \times S^p \cup D^q_2 \times S^p$ is diffeomorphic to $D^q_1 \times S^p \cup D^q_2 \times S^p$ where $\phi_i : S^{q-1} \times S^p \pm S^{q-1} \times S^p$ is the diffeomorphism defined by $\phi_i(x, y) = (\phi_i(x), y)$ and $f_b : S^{q-1} \times S^p \pm S^{q-1} \times S^p$ is defined by $f_b(x, y) = (x, b(x), y)$ where $[b] = [\beta] \in \pi_{q-1}SO(p+1)$ is the characteristic map of the bundle. The manifold $E(\Sigma^q_2)$ can be regarded as the boundary of the $(p+1)$-disc bundle over $\Sigma_2$ which is denoted by
\[ D_i^q \times D^{p+1} \cup (\phi \times \id) \cdot f_b = D(\Sigmaq^2) \]. So if \( E(\Sigmaq^1) \) is diffeomorphic to \( E(\Sigmaq^2) \) then since \( \Sigmaq^1 \) can be embedded in \( E(\Sigmaq^1) \) it follows that \( \Sigmaq^1 \) embeds in \( E(\Sigmaq^2) \). But \( \Sigmaq^2 \) naturally embeds in \( E(\Sigmaq^2) \) and so we have \( \Sigmaq^1 \) and \( \Sigmaq^2 \) sitting in \( E(\Sigmaq^2) \), if we translate \( \Sigmaq^2 \) away from \( \Sigmaq^1 \) we can run a tube between them to obtain an embedding \( \Sigmaq^1 \# (-\Sigmaq^2) \to E(\Sigmaq^2) \) so that the embedding is homotopically trivial and so by the engulfing result of [10, chapter 7] it means that \( \Sigmaq^1 \# (-\Sigmaq^2) \) can be embedded in the interior of a \( (p+q+1) \)-disc in \( E(\Sigmaq^2) \) and by [11, 3.5] the embedding is isotopic to a nuclear embedding into the interior of \( S^q \times D^{p+1} \). However the embedding \( \Sigmaq^1 \# (-\Sigmaq^2) \to S^q \times D^{p+1} \) is an homotopy equivalence, it then follows by Smale's theorem [12, Theorem 4.1] that \( \Sigmaq^1 \# (-\Sigmaq^2) \to D^{p+1} \) is diffeomorphic to \( S^q \times D^{p+1} \) so it follows that \( \Sigmaq^1 \# (-\Sigmaq^2) \times S^p \) is diffeomorphic to \( S^q \times S^p \) hence \( \Sigmaq^1 \# (-\Sigmaq^2) \) \( \in H(q,p) \). Conversely suppose \( \Sigmaq^1 \# (-\Sigmaq^2) \in H(q,p) \) then this implies \( (\Sigmaq^1 \# (-\Sigmaq^2)) \times S^p \) is diffeomorphic to \( S^q \times S^p \). Since \( S^q \times S^p \) embeds in \( R^{p+q+1} \) with trivial normal bundle then it follows that \( \Sigmaq^1 \# (-\Sigmaq^2) \) embeds in \( R^{p+q+1} \) with trivial normal bundle. This shows that each \( \Sigmaq^i \) for \( i = 1, 2 \) embeds in \( R^{p+q+1} \) with trivial normal bundle and by [11, §3.5] the embedding is isotopic to an embedding of \( \Sigmaq^i \) into the interior of \( S^q \times D^{p+1} \). However for \( i = 1, 2 \) the embedding \( \Sigmaq^i \to S^q \times D^{p+1} \) is an homotopy equivalence hence it follows from [12, Theorem 4.1] that \( \Sigmaq^1 \times D^{p+1} \) is diffeomorphic to \( S^q \times D^{p+1} \) which implies that \( \Sigmaq^1 \times D^{p+1} \) is diffeomorphic to \( \Sigmaq^2 \times D^{p+1} \). Now since \( \Sigmaq^i = D_i^q \cup D_2^q \) where \( \phi_i : S^{q-1} \to S^{q-1} \) represents \( \Sigmaq^i \in \pi_q^i \) \( \in 1, 2 \), then we can write \( \Sigmaq^i \times D^{p+1} \cup D^{q-1} \cup D_2^q \) \( \subset D^{p+1} \) where we identify two copies of \( D^q \) \( \times \) \( D^{p+1} \) along \( S^{q-1} \times D^{p+1} \) by the diffeomorphism \( \phi : S^{q-1} \times D^{p+1} \to S^{q-1} \times D^{p+1} \) defined by \( (\phi \times \id)(x,y) = (\phi(x),y) \) where \( (x,y) \in S^{q-1} \times D^{p+1} \). So \( \Sigmaq^1 \times D^{p+1} \) is diffeomorphic to \( \Sigmaq^2 \times D^{p+1} \) implies \( D_1^q \times D^{p+1} \cup D_2^q \times D^{p+1} \) \( \subset D_2^q \times D^{p+1} \) \( \subset D_2^q \times D^{p+1} \). Now consider the manifold \( D(S^q) = D_i^q \times D^{p+1} \cup D_2^q \times D^{p+1} \) which is a \( (p+1) \)-disc bundle over a \( q \)-sphere with characteristic map \( b \in \pi_{q-1} SO(p+1) \). We then form the quotient space
\[ D(S^q) \cup \Sigmaq^1 \times D^{p+1} = (D_i^q \times D^{p+1} \cup D_2^q \times D^{p+1}) \cup (D_1^q \times D^{p+1} \cup D_2^q \times D^{p+1}) \]
by identifying \( D_2^q \times D^{p+1} \subset D(S^q) \) and \( D_i^q \times D^{p+1} \subset \Sigmaq^1 \times D^{p+1} \) by the relation \( (x,y) = (x,y) \) \( (x \in D_2^q, y \in D^{p+1}) \). The manifold \( D(S^q) \cup \Sigmaq^2 \times D^{p+1} \) is similarly constructed. Since \( \Sigmaq^1 \times D^{p+1} \) is diffeomorphic to \( \Sigmaq^2 \times D^{p+1} \). Let \( d : \Sigmaq^1 \times D^{p+1} + \Sigmaq_2 \times D^{p+1} \) be the
diffeomorphism and since any diffeomorphism fixes a disc, we can assume that \( d \) is identity on the disc \( D^{p+q+1} = D^{p}_1 \times D^{q+1} \), then we can define a diffeomorphism.

\[
g : D(S^q) \cup \Sigma^q_1 \times D^{p+1} \to D(S^q) \cup \Sigma^q_2 \times D^{p+1}
\]

where

\[
g(x) = \begin{cases} 
  d(x) & \text{for } x \in \Sigma^q_1 \times D^{p+1} \\
  x & \text{for } x \in D(S^q).
\end{cases}
\]

This means that \( g = d \) on \( \Sigma^q_1 \times D^{p+1} \) and identity on \( D(S^q) \). \( g \) is well defined because \( d \) is identity on the disc connecting \( D(S^q) \) and \( \Sigma^q_1 \times D^{p+1} \) and \( g \) is a diffeomorphism. The manifold \( D(S^q) \cup \Sigma^q_1 \times D^{p+1} \) can be clearly seen as follows. Let \( (\phi_i \times \text{id}) \cdot f_b : S^{q-1} \times D^{p+1} \to S^{q-1} \times D^{p+1} \) be the diffeomorphism defined by \( ((\phi_i \times \text{id}) \cdot f_b)(x,y) = (\phi_i(x), b(x) \cdot y) \), \((x,y) \in S^{q-1} \times D^{p+1} \) then attaching two manifolds \( D^q_+ \times D^{p+1} \) and \( D^q_- \times D^{p+1} \) by the diffeomorphism \( (\phi_i \times \text{id}) \cdot f_b \) we have \( D^q_+ \times D^{p+1} \cup D^q_- \times D^{p+1} \) we get a \((p+1)\)-disc

bundle over the homotopy q-sphere \( \Sigma^q_i = D^q_i \cup D^q_{i+1} \) \( i = 1, 2 \). However, from the way \( (\phi_i \times \text{id}) \cdot f_b \) is constructed it is easily seen that \( D(S^q) \cup \Sigma^q_1 \times D^{p+1} = D^q_+ \times D^{p+1} \cup D^q_- \times D^{p+1} = D(S^q) \) hence \( g \) is the diffeomorphism of \( D(S^q) \) onto \( D(S^q) \).

Then it follows that \( \partial(D(\Sigma^q_1)) = E(\Sigma^q_1) \) is diffeomorphic to \( \partial(D(\Sigma^q_2)) = E(\Sigma^q_2) \).

Hence the theorem is proved.

REMARK 1. This theorem implies that \( E(\Sigma^q_1) \) is diffeomorphic to \( E(\Sigma^q_2) \) if and only if \( \Sigma^q_1 \) and \( \Sigma^q_2 \) are equivalent in the quotient group \( \theta^q/H(q,p) \).

To complete this classification, we need to determine the inertial group of \( E(\Sigma^q) \). The inertial group \( \Theta(M) \) of an oriented closed smooth \( n \)-dimensional manifold \( M \) is defined to be the subgroup of \( \Theta^n \) consisting of those homotopy \( n \)-spheres \( \Sigma^n \) such that \( M \# \Sigma^n \) is diffeomorphic to \( M \).

Let \( E_\beta \) represent the total space of a \( p \)-sphere bundle over a real \( q \)-sphere with characteristic class \( \beta \in \pi_{q-1}SO(p+1) \). In [13] we defined a map \( G_\beta : \pi_pSO(q) \to \theta^q \) and showed that the image of this map equals the inertial group of \( E_\beta \) where \( p < q \) and \( E_\beta \) has no cross-section. We shall similarly define a map \( G_{\phi \beta} : \pi_pSO(q) \to \theta^{p+q} \) and show that the image of this map equals the inertial group of \( E(\Sigma^q) \) where \( E(\Sigma^q) \) is the total space of a \( p \)-sphere bundle over a homotopy sphere \( \Sigma^q = D^q_1 \cup D^q_2 \). Let \( \alpha \in \pi_pSO(q) \) we define

\[
G_{\phi \beta}(a) = S^{q-1} \times D^{p+1} \cup \_f_{\phi^{-1}(\phi \times \text{id}) \cdot f_b} D^q \times S^p \text{ where } [a] = \alpha \text{ and } [b] = \beta \in \pi_{q-1}SO(p+1) \text{ and }
\]
$f_{a-1}(\phi\times\text{id})\cdot f_b : S^{q-1} \times D^p \to S^{q-1} \times D^p$ is a diffeomorphism defined by $f_{a-1}(\phi\times\text{id})\cdot f_b(x,y) = (a^{-1}(b(x)\cdot y) \cdot (x), b(x)\cdot y)$. One can easily show that $G_{\phi,q}$ is well-defined and that its image is an homotopy $(p+q)$-sphere as similarly shown in [13].

**Lemma 3.2.** Let $E(\Sigma^q)$ denote the total space of a $p$-sphere bundle over an homotopy $q$-sphere $\Sigma^q = D_1^q \cup_{\partial D_2^q} D_2^q$ with characteristic class $\beta \in \pi_{q-1}SO(p+1)$ then $G_{\phi,q}^\ast\pi_{p}SO(q)) \cong (E(\Sigma^q))$.

**Proof.** If $E^{p+q} \subset I(E(\Sigma^q))$ then this means there is a diffeomorphism $d : E(\Sigma^q) \# \Sigma^{p+q} \to E(\Sigma^q)$, that is,

$$d : (D_1^q \times D^p) \cup (\phi\times\text{id})\cdot f_b D_2^q \times D^p \# \Sigma^{p+q} \to D_1^q \times D^p \cup (\phi\times\text{id})\cdot f_b D_2^q \times D^p$$

since $p < q$ then $\pi_p(E(\Sigma^q))$ is infinitely cyclic and $d(\phi\times\text{id})\cdot f_b$ represents a generator and so is homotopic to the inclusion $0 \times D^p \to E(\Sigma^q)$. By Haefliger's theorem [14], $d|0 \times D^p$ and the inclusion $0 \times D^p \to E(\Sigma^q)$ are isotopic and by isotopy extension theorem and tubular neighborhood theorem, $d$ is isotopic to a map which we shall again denote by $d$ such that $d|D^q \times D^p = D^q \times D^p$ where $d(x,y) = (a(y)\cdot x, y)$ for $a \in \pi_pSO(q)$ and $(x,y) \in D^q \times D^p$. We now remove $D^q \times D^p$ from $E(\Sigma^q) \# \Sigma^{p+q} = (D^q \times D^p \cup D^q \times D^p) \# \Sigma^{p+q}$ by $(\phi\times\text{id})\cdot f_b$ surgery away from the connected sum and replace it with $S^{p+1} \times D^p$. After this operation on the summand $E(\Sigma^q)$ of the connected sum, we have the manifold $S^{q-1} \times D^{p+1}$.

$$U D^q \times D^p \cup (\phi\times\text{id})\cdot f_b$$

Since the diffeomorphism $(\phi\times\text{id})\cdot f_b : S^{q-1} \times D^p \to S^{q-1} \times D^p$ extend to the diffeomorphism of $S^{q-1} \times D^{p+1}$ onto itself then $S^{q-1} \times D^{p+1} \cup (\phi\times\text{id})\cdot f_b D^q \times D^p$ is diffeomorphic to $S^{q-1} \times D^{p+1} \cup (\phi\times\text{id})\cdot f_b D^q \times D^p$, the diffeomorphism $g$ is defined thus

$$g(x,y) = \begin{cases} (x,y) & \text{if } (x,y) \in D^q \times D^p \\ ((\phi\times\text{id})\cdot f_b)(x,y) & \text{if } (x,y) \in S^{q-1} \times D^{p+1}. \end{cases}$$

However, by [7, Lemma 2.1.2], $S^{q-1} \times D^{p+1} \cup (\phi\times\text{id})\cdot f_b D^q \times D^p$ is diffeomorphic to the standard $(p+q)$-sphere $\Sigma^{p+q}$, hence after this surgery $E(\Sigma^q)$ is reduced to $S^{p+d}$ and so $E(\Sigma^q) \# \Sigma^{p+q}$ is reduced to $S^{p+q} \times \Sigma^{p+q} = \Sigma^{p+q}$.
We perform the corresponding modification (under d) on \( E(\mathbb{Z}^q) \) to remove the p-sphere \( 0 \times S^p \) with product structure \( d(D_1^q \times S^p) \) in \( E(\mathbb{Z}^q) \). From this modification we obtain a manifold \( S^{q-1} \times D^{p+1} \cup D^q \times S^p \) where \( \psi = (d^{-1}|S^{q-1} \times S^p) \cdot (\psi \cdot \text{id}) \cdot f_b \) and this is diffeomorphic to \( S^{p+q} \) because of the way we performed the surgery using d. However, this manifold \( S^{q-1} \times D^{p+1} \cup D^q \times S^p = G_{\phi \cdot \beta}(\alpha) \) by the definition of \( G_{\phi \cdot \beta} \), thus there exists an element \( \alpha \in \pi_p \text{SO}(q) \) (namely) \( d(D_1^q \times S^p) \) which gives \( \alpha \in \pi_p \text{SO}(q) \) such that \( S^{p+q} = G_{\phi \cdot \beta}(\alpha) \) and so \( S^{p+q} \in G_{\phi \cdot \beta}(\pi_p \text{SO}(q)) \), hence \( I(E(\mathbb{Z}^q)) \subseteq G_{\phi \cdot \beta}(\pi_p \text{SO}(q)) \). Conversely suppose \( S^{p+q} \in G_{\phi \cdot \beta}(\pi_p \text{SO}(q)) \) then for some \( \alpha \in \pi_p \text{SO}(q) \), \( S^{p+q} = S^{q-1} \times D^{p+1} \cup f_{a^{-1}}(\psi \cdot \text{id}) \cdot f_b \).

\( D^q \times S^p \) where \( \phi \) is a diffeomorphism of \( S^{q-1} \) onto itself representing \( \mathbb{Z}^q = D_1^q \cup D_2^q \) and \( f_{a^{-1}} \) and \( f_b \) are as defined earlier. Notice that \( G_{\phi \cdot \beta}(\alpha) \) is thus the obstruction to the construction of a diffeomorphism \( S^{p+q} \to S^{p+q} \). To construct a diffeomorphism from \( S^{p+q} \) to \( S^{p+q} \), we map \( S^{q-1} \times D^{p+1} \subset S^{p+q} \) to itself using \( (\psi \cdot \text{id}) \cdot f_b \) to have

\[
S^{p+q} = S^{q-1} \times D^{p+1} \cup D^q \times S^p
\]

\[
S^{p+q} = S^{q-1} \times D^{p+1} \cup f_{a^{-1}}(\psi \cdot \text{id}) \cdot f_b
\]

and try to extend it to \( D^q \times S^p \). On the boundary \( S^{q-1} \times S^p \) of \( D^q \times S^p \), the map is \( f_{b^{-1}}(\psi \cdot \text{id}) \cdot f_{a^{-1}}(\psi \cdot \text{id}) \cdot f_b \). So this means that \( S^{p+q} = G_{\phi \cdot \beta}(\alpha) \) is the obstruction to extending the diffeomorphism \( f_{b^{-1}}(\psi \cdot \text{id}) \cdot f_{a^{-1}}(\psi \cdot \text{id}) \cdot f_b : S^{q-1} \times S^p \to S^{q-1} \times S^p \) to a diffeomorphism of \( D^q \times S^p \) onto itself. We can then define a map \( E(\mathbb{Z}^q) \to E(\mathbb{Z}^q) \) using the diffeomorphism \( f_a : D_1^q \times S^p \to D_1^q \times S^p \) where \( f_a(x,y) = (\alpha(y),x,y) \) \((x,y) \in D_1^q \times S^p \) we then have

\[
E(\mathbb{Z}^q) = D_1^q \times S^p \cup D_2^q \times S^p
\]

\[
E(\mathbb{Z}^q) = D_1^q \times S^p \cup (\psi \cdot \text{id}) \cdot f_b
\]

\[
E(\mathbb{Z}^q) = D_1^q \times S^p \cup D_2^q \times S^p
\]

On the boundary \( S^{q-1} \times S^p \) of \( D_1^q \times S^p \), this map is \( f_{b^{-1}}(\psi \cdot \text{id}) \cdot f_{a^{-1}}(\psi \cdot \text{id}) \cdot f_b \) and the obstruction to extending this to a diffeomorphism of \( E(\mathbb{Z}^q) \) onto itself is the
obstruction to extending the map $f_{b^{-1}}(\phi^{-1} \times \text{id}) \cdot f_{b}(\phi \times \text{id}) \cdot f_{b}$ to the diffeomorphism of $D^p_2 \times S^p$ onto itself which is $\varepsilon^{p+q}$. It then follows that $E(\varepsilon^q) \cdot E(\varepsilon^q) \neq \varepsilon^{p+q}$ is a diffeomorphism and so $\varepsilon^{p+q} \in I(E(\varepsilon^q))$ hence

$$E(E(\varepsilon^q)) = G \cdot \beta_{p}^{n}(SO(q))$$

REMARK 2. We note that if $p \equiv 2, 4, 5, 6 \pmod{8}$ and $p < q-1$ then $\pi_p SO(q) = 0$ and so the image of $G$ is trivial and hence in this particular case, the inertial group of $E(\varepsilon^q)$ is trivial and this coincides with the result of [4, Proposition 1].

REMARK 3. By [15], inertial group $I(M)$ of a smooth manifold $M$ is a diffeotopy invariant of $M$. So if $2p \geq q+1$ then we can deduce that the inertial group $I(E(\varepsilon^q))$ of a $p$-sphere bundle over an homotopy $q$-sphere $\varepsilon^q$ is equal to the inertial group $I(E_B)$ of a $p$-sphere bundle over the standard $q$-sphere, where $B \in \Pi_{q-1} SO(p+1)$ classifies the associated disc bundle. Let $D(\varepsilon^q)$ be the associated $(p+1)$-disc bundle over the homotopy $q$-sphere where $E(\varepsilon^q)$ is the boundary of $D(\varepsilon^q)$. $\varepsilon^q$ has the homotopy type of $D(\varepsilon^q)$ and $\varepsilon^q$ has the homotopy type of $S^q$, it follows that $S^q$ has the homotopy type of $D(\varepsilon^q)$. Since $2p \geq q+1$ then it follows that $2(p+q+1) \geq 3q + 3$ and since $p + q > 5$ and $p \geq 3$ then $D(\varepsilon^q)$ and $E(\varepsilon^q)$ are simply connected and from [12: Theorem 4.4], it follows that $D(\varepsilon^q)$ is diffeomorphic to a $(p+1)$-disc bundle $D(S^q)$ over the $q$-sphere $S^q$ hence the boundary $\partial D(\varepsilon^q) = E(\varepsilon^q)$ of $D(\varepsilon^q)$ is diffeomorphic to the boundary $\partial D(S^q) = E_B$ of $D(S^q)$. It then follows by [15] that $I(E(\varepsilon^q)) = I(E_B)$. This means that the inertial group of $S_B$ in [13] coincides with Lemma 3.2.

Combination of Lemmas 3.1 and 3.2 give the following.

THEOREM 3.3. Let $E$ be the total space of a $p$-sphere bundle over a $q$-sphere with characteristic map $\beta \in \Pi_{q-1} SO(p+1)$ then the diffeomorphism classes of $p+q$-manifolds that are homeomorphic to $E$ are in one-to-one correspondence with the group

$$\frac{\mathbb{H}(q,p) \times \mathbb{B}_n}{\text{Image } G_B}$$

where $p+q = n \geq 6$ and $p < q$.

REFERENCES


