MIXED FOLIATE CR-SUBMANIFOLDS IN A COMPLEX HYPERBOLIC SPACE ARE NON-PROPER

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ABSTRACT. It was conjectured in [1 II] (also in [2]) that mixed foliate CR-submanifolds in a complex hyperbolic space are either complex submanifolds or totally real submanifolds. In this paper we give an affirmative solution to this conjecture.

KEY WORDS AND PHRASES: CR-submanifolds, complex hyperbolic space, mixed foliate.

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1. INTRODUCTION.

A submanifold M of a Kaehler manifold $\tilde{M}$ is called a *CR-submanifold* if (1) the maximal complex subspace $\mathcal{D}_x$ of the tangent space $T_x\tilde{M}$ containing in T_xM , $x \in M$, defines a differentiable distribution $\mathcal{D}$, called the holomorphic distribution, and (2) the orthogonal complementary distribution $\mathcal{D}^\perp$ of $\mathcal{D}$ in TM is a totally real distribution, i.e., $J\mathcal{D}_x^\perp \subset T_x^\perp M$, where J denotes the almost complex structure of $\tilde{M}$ and $T_x^\perp M$ the normal space of M at x . Complex submanifolds and totally real submanifolds of $\tilde{M}$ are trivial examples of CR-submanifolds. A CR-submanifold is called *proper* if it is neither a complex submanifold nor a totally real submanifold. The totally real distribution $\mathcal{D}^\perp$ of a CR-submanifold of a Kaehler manifold is always integrable [1,3]. A CR-submanifold M is called *mixed foliate* if (a) the holomorphic distribution $\mathcal{D}$ is integrable, and (b) the second fundamental form σ^0 of M in $\tilde{M}$ satisfies $\sigma^0(\mathcal{D}, \mathcal{D}^\perp) = \{0\}$.

It is known that mixed foliate CR-submanifolds in $\mathbb{C}^m$ are exactly CR-products in $\mathbb{C}^m$ [1 I] and mixed foliate CR-submanifolds in $\mathbb{C}P^m$ are non-proper [4]. It was conjectured in [1 II] (also in [2]) that mixed foliate CR-submanifolds in a complex hyperbolic space H^m are non-proper too.

In this paper, we solve this conjecture completely to give the following

THEOREM 1. *Let M be a mixed foliate CR-submanifold of H^m . Then M is either a complex submanifold or a totally real submanifold.*

2. PRELIMINARIES.

For simplicity, we assume that H^m is the (complex) m -dimensional complex hyperbolic space with constant holomorphic sectional curvature -4 . Let M be a mixed foliate CR-submanifold of H^m . Then, by definition, the holomorphic distribution $\mathcal{D}$ of M is integrable and the second fundamental form σ° of M in H^m satisfies $\sigma^\circ(\mathcal{D}, \mathcal{D}^\perp) = \{0\}$. We denote by $\langle \cdot, \cdot \rangle$ the metric tensor of H^m as well as the induced one on M . Let D° and A° denote the normal connection and the Weingarten map of M in H^m , respectively. If N is a leaf of $\mathcal{D}$, then N is a complex submanifold of H^m . Denote by σ, D, A and ∇ the second fundamental form, the normal connection, the Weingarten map and the Levi-Civita connection of N (in H^m), respectively, and by σ', D', A' the corresponding quantities for N in M . Then we have $\sigma(X, Y) = \sigma'(X, Y) + \sigma^\circ(X, Y)$ for X, Y tangent to N . Since $\sigma^\circ(\mathcal{D}, \mathcal{D}^\perp) = \{0\}$, we also have $A_J Z = A'_J Z$, on TN , for Z in $\mathcal{D}^\perp$. Since N is a complex submanifold of H^m , the almost complex structure J satisfies $\sigma(JX, Y) = J\sigma(X, Y) = \sigma(X, JY)$, $A_J \xi = JA\xi$, $JA\xi = -A_\xi J$, for X, Y tangent to N and ξ normal to N .

For any vector X tangent to M , we put $JX = PX + FX$ where PX and FX are the tangential and the normal components of JX , respectively. For a vector ξ normal to M , we put $J\xi = t\xi + f\xi$, where $t\xi$ and $f\xi$ are the tangential and the normal components of $J\xi$, respectively.

Since H^m is of constant holomorphic sectional curvature -4 , the curvature tensor $\tilde{R}$ of H^m is given by

$$\begin{aligned} \tilde{R}(X, Y)Z &= \langle X, Z \rangle Y - \langle Y, Z \rangle X + \langle JX, Z \rangle JY \\ &\quad - \langle JY, Z \rangle JX - 2\langle X, JY \rangle JZ \end{aligned} \quad (2.1)$$

for X, Y, Z tangent to H^m .

We need the following result of [1 II] for later use.

LEMMA 1. *Let M be a mixed foliate CR-submanifold of H^m . Then*

- (a) $D_X JZ = D_X^\circ JZ = F\nabla_X^\circ Z$, (b) $D_X Z = D_X' Z = -tD_X^\circ JZ$, (c) $\text{Im } \sigma = \mathcal{D}^\perp \oplus J\mathcal{D}^\perp$,
(d) $A_Z, A_{JZ} \in 0(2h)$, and (e) $A_Z A_W + A_W A_Z = 0$, for X tangent to N and orthonormal vectors Z and W in $\mathcal{D}^\perp$.

LEMMA 2. *Under the hypothesis of Lemma 1, if M is proper, then (a) each leaf N of $\mathcal{D}$ lies in a complex $(h+p)$ -dimensional totally geodesic complex submanifold H^{h+p} of H^m and (b) $h+1 \geq p \geq 2$ and $h \geq 2$ where $p = \dim_{\mathbb{R}} \mathcal{D}^\perp$ and $h = \dim_{\mathbb{R}} \mathcal{D}$.*

3. MORE LEMMAS.

Let M be a mixed foliate CR-submanifold of H^m . If M is non-proper, there is nothing to prove. Thus we may assume that M is proper. By Lemma 2, $p \geq 2$. From Lemma 1, we have

$$A_Z A_W + A_W A_Z = 0 \quad (3.1)$$

for orthonormal vectors Z, W in $\mathcal{D}^\perp$. Let $Z_1, \dots, Z_p$ be an orthonormal frame of $\mathcal{D}^\perp$. We put

$$A_\alpha = A_{Z_\alpha}, \quad A_{\alpha*} = A_{JZ_\alpha}, \quad \alpha = 1, \dots, p. \quad (3.2)$$

From property (d) of Lemma 1, each $A_{\alpha*}$ has eigenvalues 1 and -1 with the same multiplicity h . Let $X_1, \dots, X_h$ be h orthonormal eigenvectors of $A_{\alpha*}$ with eigenvalue 1. Then $JX_1, \dots, JX_h$ are eigenvectors of $A_{\alpha*}$ with eigenvalue -1. With respect to the basis $\{X_1, \dots, X_h, JX_1, \dots, JX_h\}$, we have

$$A_{\alpha*} = \begin{pmatrix} I_h & 0 \\ 0 & -I_h \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -I_h \\ I_h & 0 \end{pmatrix}, \quad (3.3)$$

where I_h denotes the $h \times h$ identity matrix. Thus, by (2.1), we have

$$A_\alpha = \begin{pmatrix} 0 & -I_h \\ -I_h & 0 \end{pmatrix}. \quad (3.4)$$

In particular, if we choose $\alpha = 1$, we obtain

$$A_1 = \begin{pmatrix} 0 & -I_h \\ -I_h & 0 \end{pmatrix}, \quad A_{1*} = \begin{pmatrix} I_h & 0 \\ 0 & -I_h \end{pmatrix}. \quad (3.5)$$

From (2.1) and (3.1) we have

$$A_\alpha A_{\beta*} - A_{\beta*} A_\alpha = 0, \quad \alpha \neq \beta, \quad \alpha, \beta = 1, \dots, p. \quad (3.6)$$

Using (3.1), (3.5) and (3.6) we may get

$$A_2 = \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}, \quad A_{2*} = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}. \quad (3.7)$$

Since $A_2 \in O(2h)$ (Lemma 1), we also have

$$B \in O(h), \quad {}^t B = B, \quad (3.8)$$

where ${}^t B$ denotes the transpose of B .

LEMMA 3. If M is a proper mixed foliate CR-submanifold of H^m , then $p \geq 3$.

PROOF. Under the hypothesis, Lemma 2 shows that if $p < 3$, then $p = 2$. If $p = 2$, we may choose an orthonormal frame $X_1, \dots, X_h, JX_1, \dots, JX_h, Z_1, Z_2, JZ_1, JZ_2$ such that, with respect to this frame, A_1, A_2, A_{1*} and A_{2*} take the forms of (3.5), (3.7) and (3.8).

We put

$$V = \text{Span}\{X_1, \dots, X_h\}. \quad (3.9)$$

Then $TN = V \oplus JV$. Since $B \in O(h)$ with ${}^t B = B$, we may further choose $\{X_1, \dots, X_h\}$ such that with respect to it, B has the form:

$$B = \begin{pmatrix} I_r & 0 \\ 0 & -I_{h-r} \end{pmatrix} \quad (3.10)$$

for some r , $0 \leq r \leq h$.

CASE 1: $r = h$. In this case we have

$$-A_1 = A_2^* = \begin{bmatrix} 0 & I_h \\ I_h & 0 \end{bmatrix}, \quad A_1^* = A_2 = \begin{bmatrix} I_h & 0 \\ 0 & -I_h \end{bmatrix}. \quad (3.11)$$

So, if we put

$$W = \frac{1}{\sqrt{2}} (Z_1 + JZ_2), \quad (3.12)$$

then $A_W = A_{JW} = 0$, which contradicts statement (c) of Lemma 1.

CASE 2: $r = 0$. This case is impossible by applying an argument similar to Case 1.

CASE 3: $r > 0$ and $h > r$. In this case we can decompose V and JV into orthogonal decompositions:

$$V = V' \oplus V'', \quad JV = JV' \oplus JV'', \quad (3.13)$$

where V' and V'' are eigenspaces of B (defined by (3.10)) with eigenvalues 1 and -1 , respectively. By (3.5), (3.7), (3.10) and Lemma 1 we have

$$\begin{aligned} \sigma(X, T) &= \langle JX, T \rangle (JZ_2 - Z_1) + \langle X, T \rangle (JZ_1 + Z_2), \\ \sigma(Y, T) &= -\langle JY, T \rangle (JZ_2 + Z_1) + \langle Y, T \rangle (JZ_1 - Z_2) \end{aligned} \quad (3.14)$$

for $X \in V'$, $Y \in V''$ and $T \in TN$.

By Lemma 1 we have

$$DZ_1 = \lambda Z_2, \quad DZ_2 = -\lambda Z_1, \quad DJZ_1 = \lambda JZ_2, \quad DJZ_2 = -\lambda JZ_1, \quad (3.15)$$

for some 1-form λ on N . Since N is a complex submanifold of H^m , the equation of Codazzi gives

$$(\bar{\nabla}_X \sigma)(Y, Z) = (\bar{\nabla}_Y \sigma)(X, Z) \quad (3.16)$$

where $(\bar{\nabla}_X \sigma)(Y, Z) = D_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$ for X, Y, Z tangent to N .

In particular, if $X \in V'$, $Y \in V''$ and $W \in JV'$, then by applying (3.14), (3.15) and (3.16), we see that the Z_2 -components of both sides of (3.16) yield

$$0 = \lambda(Y) \langle JX, W \rangle - \langle W, \nabla_Y X \rangle + \langle X, \nabla_Y W \rangle. \quad (3.17)$$

Because $\langle X, W \rangle = 0$, (3.17) implies

$$2 \langle \nabla_Y X, W \rangle = \lambda(Y) \langle JX, W \rangle. \quad (3.18)$$

Similarly, if $X \in V'$, $Y \in V''$ and $W \in JV'$, the JZ_1 -components yield

$$2 \langle \nabla_Y X, W \rangle - \lambda(Y) \langle JX, W \rangle = 2 \langle \nabla_X Y, W \rangle. \quad (3.19)$$

Combining (3.18) and (3.19) we find

$$\langle \nabla_X Y, W \rangle = 0 \quad (3.20)$$

which also implies $\langle \nabla_X W, Y \rangle = 0$. Therefore

$$\nabla_Y V'' \perp JV', \quad \nabla_Y JV' \perp V''. \quad (3.21)$$

Since J is parallel, this also gives

$$\nabla_Y JV'' \perp V', \quad \nabla_Y V' \perp JV''. \quad (3.22)$$

Similarly, we may obtain

$$\nabla_{V'} V' \perp V'', \quad \nabla_{V'} JV' \perp JV'', \quad (3.23)$$

$$\nabla_{V'} V'' \perp V', \quad \nabla_{V'} JV'' \perp JV'. \quad (3.24)$$

Let $U' = V' \oplus JV'$ and $U'' = V'' \oplus JV''$. Then (3.21) - (3.24) show that

$$\nabla_{V'} U' \perp U'', \quad \nabla_{V'} U'' \perp U'. \quad (3.25)$$

In a similar way we may also obtain $\nabla_{JV'} U' \perp U''$ and $\nabla_{JV'} U'' \perp U'$. Therefore, we see that U' and U'' are both integrable and parallel distributions. Thus N is locally the Riemannian product of two Kaehler manifolds. This is a contradiction since H^m admits no complex submanifold which is a product of two Kaehler manifolds (cf. [1 I]). (Q.E.D.)

LEMMA 4. Let M be a proper mixed foliate CR-submanifold of H^m . If $p = \dim_{\mathbb{R}} D^\perp \geq 3$, then $h = \dim_{\mathbb{C}} D = 2r$ is even and with respect to a suitable orthonormal frame $X_1, \dots, X_h, JX_1, \dots, JX_h, Z_1, \dots, Z_p, JZ_1, \dots, JZ_p$, we have

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & -I_h \\ -I_h & 0 \end{pmatrix}, & A_{1*} &= \begin{pmatrix} I_h & 0 \\ 0 & -I_h \end{pmatrix}, \\ A_2 &= \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}, & A_{2*} &= \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}, & B &= \begin{pmatrix} I_r & 0 \\ 0 & -I_r \end{pmatrix}, \\ A_3 &= \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}, & A_{3*} &= \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, & C &= \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}. \end{aligned} \quad (3.27)$$

If $p \geq 4$, then, for $\alpha \geq 4$, we also have

$$A_\alpha = \begin{pmatrix} D_\alpha & 0 \\ 0 & -D_\alpha \end{pmatrix}, \quad A_{\alpha*} = \begin{pmatrix} 0 & D_\alpha \\ D_\alpha & 0 \end{pmatrix}, \quad D_\alpha = \begin{pmatrix} 0 & E_\alpha \\ {}^t E_\alpha & 0 \end{pmatrix} \quad (3.28)$$

for some $E_\alpha \in O(r)$ such that ${}^t E_\alpha = -E_\alpha$.

PROOF. Under the hypothesis, there is a suitable orthonormal frame $X_1, \dots, X_h, JX_1, \dots, JX_h, Z_1, \dots, Z_p, JZ_1, \dots, JZ_p$ such that A_1, A_2, A_{1*} and A_{2*} take the desired forms (cf. (3.5), (3.7) and (3.10)). Since $A_\alpha A_1 + A_1 A_\alpha = 0$ for $\alpha \geq 3$, we also have

$$A_\alpha = \begin{pmatrix} D_\alpha & 0 \\ 0 & -D_\alpha \end{pmatrix}, \quad A_{\alpha*} = \begin{pmatrix} 0 & D_\alpha \\ D_\alpha & 0 \end{pmatrix}, \quad (3.29)$$

where $D_\alpha \in O(h)$ with ${}^t D_\alpha = D_\alpha$. From Lemma 1 we also have

$$A_2 A_\alpha + A_\alpha A_2 = 0, \quad A_2 A_{\alpha*} - A_{\alpha*} A_2 = 0. \quad (3.30)$$

From this we see that each D_α takes the following form:

$$D_\alpha = \begin{pmatrix} 0 & E_\alpha \\ {}^t E_\alpha & 0 \end{pmatrix}, \quad \alpha \geq 3, \quad (3.31)$$

where each E_α is a $(r \times (h-r))$ -matrix. Since $D_\alpha \in O(h)$, this implies

$$E_{\alpha}^t E_{\alpha} = I_r \quad \text{and} \quad {}^t E_{\alpha} E_{\alpha} = I_{h-r}. \quad (3.32)$$

It is clear that this is impossible unless E_{α} is a square matrix. Therefore, we have $r = 0$, $h = r$, or $h = 2r$. However, the first two cases cannot occur since, for instance, if $r = 0$, then $A_2 = -A_1^*$ which implies $A_{\alpha} = 0$ by virtue of (3.30). This contradicts to (c) of Lemma 1. Similar argument works for the second case. Consequently, $h = 2r$ which is even. Now, let $X_1, \dots, X_h$ be chosen in such a way that

$$X_{r+1} = A_3 X_1, \dots, X_h = A_3 X_r.$$

Then A_3 and A_3^* are expressed in the forms given in (3.31). Finally, for each $\alpha \geq 4$, by using the properties $A_3 A_{\alpha} + A_{\alpha} A_3 = 0$ and $D_{\alpha} \in 0(h)$, we may conclude that D_{α} is in the desired form. (Q.E.D.)

LEMMA 5. Let M be a proper mixed foliate CR-submanifold of H^m . If $p \geq 4$, then $h \geq 2p-4$. Furthermore, we may choose the orthonormal frame such that, in addition to (3.27) and (3.28), we also have

$$A_{\alpha} A_3 X_1 = X_{\alpha-2}, \quad A_{\alpha} X_1 = -X_{r+\alpha-2}, \quad p \geq \alpha \geq 4, \quad (3.33)$$

$$Y_i = X_{r+i} = A_3 X_i, \quad i = 1, \dots, r. \quad (3.34)$$

PROOF. As given in the proof of Lemma 3, we decompose the tangent bundle of N into orthogonal decomposition:

$$TN = V \oplus JV, \quad V = V' \oplus V'', \quad JV = JV' \oplus JV''. \quad (3.35)$$

Such a decomposition is given with respect to A_1^* and A_2 . Now, let X_1 be a unit vector in V' . We put $Y_1 = X_{r+1} = A_3 X_1$ as before. Then (e) of Lemma 1 implies that $A_3 Y_1, \dots, A_p Y_1$ are orthonormal vectors in V' (cf. p. 500 of [4 II]). From this we conclude that $r \geq p-2$ which is equivalent to $h \geq 2p-4$. Now, we put

$$X_i = A_{i+2} A_3 X_1 = A_{i+2} Y_1, \quad 2 \leq i \leq 2, \quad (3.36)$$

$$Y_i = X_{r+i} = A_3 X_i, \quad \text{for } i = 2, \dots, p-2, \dots, r. \quad (3.37)$$

Then, (3.27) holds. Since

$$A_{\alpha} X_1 = A_{\alpha} A_3 Y_1 = -A_3 A_{\alpha} Y_1 = -A_3 X_{\alpha-2} = -Y_{\alpha-2}, \quad (3.38)$$

we also have (3.33). Formulas (3.34) are nothing but (3.37). (Q.E.D.)

From properties (a) and (b) of Lemma 1, we have

$$DZ_{\alpha} = \sum_{\beta=1}^p \theta_{\alpha\beta} Z_{\beta}, \quad \theta_{\alpha\beta} = -\theta_{\beta\alpha}, \quad \alpha, \beta = 1, \dots, p. \quad (3.39)$$

for some 1-forms $\theta_{\alpha\beta}$ on N . (3.39) gives

$$DJZ_{\alpha} = \sum_{\beta} \theta_{\alpha\beta} JZ_{\beta}. \quad (3.40)$$

LEMMA 6. Under the hypothesis and the notations of Lemma 5, we have

$$2\langle \nabla_T X_j, JX_k \rangle = \delta_{jk} \theta_{1,2}(T), \quad (3.41)$$

$$2\langle \nabla_T Y_j, JY_k \rangle = \delta_{jk} \theta_{2,1}(T), \quad (3.42)$$

$$2\langle \nabla_T X_j, JY_k \rangle = \delta_{jk} \theta_{1,3}(T) + \sum_{\alpha \geq 4} \langle A_{\alpha} X_j, Y_k \rangle \theta_{1,\alpha}(T), \quad (3.43)$$

$$2\langle \nabla_T X_j, Y_k \rangle = \delta_{jk} \theta_{23}(T) + \sum_{\alpha \neq 4} \langle A_\alpha X_j, Y_k \rangle \theta_{2\alpha}(T), \quad (3.44)$$

$$\langle \nabla_T Y_i, Y_k \rangle - \langle \nabla_T X_i, X_k \rangle = \sum_{\alpha \neq 4} \langle A_\alpha X_i, Y_k \rangle \theta_{3\alpha}(T) \quad (3.45)$$

for T tangent to N .

PROOF. The proof of this lemma is based mainly on the equation of Codazzi. Let $X_1, \dots, X_r, Y_1, \dots, Y_r$ be an orthonormal frame of $V' \oplus V'' = V$ with $Y_i = X_{r+i} = A_3 X_i$ as before, then for any vector T tangent to N , Lemma 4 gives

$$\sigma(X_i, T) = \langle JX_i, T \rangle (JZ_2 - Z_1) + \langle X_i, T \rangle (Z_2 + JZ_1) \quad (3.46)$$

$$+ \langle Y_i, T \rangle Z_3 + \langle JY_i, T \rangle JZ_3 + \sum_{\alpha \neq 4} (\langle A_\alpha X_i, T \rangle Z_\alpha + \langle A_{\alpha*} X_i, T \rangle JZ_\alpha),$$

$$\begin{aligned} \sigma(Y_i, T) &= -\langle JY_i, T \rangle (JZ_2 + Z_1) - \langle Y_i, T \rangle (Z_2 - JZ_1) \\ &+ \langle X_i, T \rangle Z_3 + \langle JX_i, T \rangle JZ_3 + \sum_{\alpha \neq 4} (\langle A_\alpha Y_i, T \rangle Z_\alpha + \langle A_{\alpha*} Y_i, T \rangle JZ_\alpha). \end{aligned}$$

From (3.46), (3.47), (2.3) and Lemmas 4 and 5, we obtain

$$\begin{aligned} (\bar{\nabla}_{X_i} \sigma)(JY_j, JY_k) &= D_{X_i}(\delta_{jk} Z_2 - \delta_{jk} JZ_1) - \langle JY_k, \nabla_{X_i} Y_j \rangle (JZ_2 + Z_1) \\ &- \langle Y_k, \nabla_{X_i} Y_j \rangle (Z_2 - JZ_1) + \langle X_k, \nabla_{X_i} Y_j \rangle Z_3 + \langle JX_k, \nabla_{X_i} Y_j \rangle JZ_3 \\ &+ \sum_{\alpha \neq 4} (\langle A_\alpha Y_k, \nabla_{X_i} Y_j \rangle Z_\alpha + \langle A_{\alpha*} Y_k, \nabla_{X_i} Y_j \rangle JZ_\alpha) \\ &- \langle JY_j, \nabla_{X_i} Y_k \rangle (JZ_2 + Z_1) - \langle Y_j, \nabla_{X_i} Y_k \rangle (Z_2 - JZ_1) \\ &+ \langle X_j, \nabla_{X_i} Y_k \rangle Z_3 + \langle JX_j, \nabla_{X_i} Y_k \rangle JZ_3 \\ &+ \sum_{\alpha \neq 4} (\langle A_\alpha Y_j, \nabla_{X_i} Y_k \rangle Z_\alpha + \langle A_{\alpha*} Y_j, \nabla_{X_i} Y_k \rangle JZ_\alpha). \end{aligned} \quad (3.48)$$

Moreover, from (3.46), (3.47) and Lemmas 4 and 5, we also obtain

$$\begin{aligned} (\bar{\nabla}_{JY_j} \sigma)(X_i, JY_k) &= D_{JY_j}(\delta_{ik} JZ_3 + \sum_{\alpha \neq 4} \langle A_\alpha X_i, Y_k \rangle JZ_\alpha) \\ &+ \langle Y_k, \nabla_{JY_j} X_i \rangle (JZ_2 + Z_1) + \langle Y_k, \nabla_{JY_j} JX_i \rangle (Z_2 - JZ_1) \\ &- \langle X_k, \nabla_{JY_j} JX_i \rangle Z_3 - \langle X_k, \nabla_{JY_j} X_i \rangle JZ_3 \\ &- \sum_{\alpha \neq 4} (\langle A_\alpha Y_k, \nabla_{JY_j} JX_i \rangle Z_\alpha + \langle A_{\alpha*} Y_k, \nabla_{JY_j} JX_i \rangle JZ_\alpha) \\ &- \langle X_i, \nabla_{JY_j} Y_k \rangle (JZ_2 - Z_1) - \langle X_i, \nabla_{JY_j} JY_k \rangle (Z_2 + JZ_1) \\ &- \langle Y_i, \nabla_{JY_j} JY_k \rangle Z_3 - \langle Y_i, \nabla_{JY_j} Y_k \rangle JZ_3 \\ &- \sum_{\alpha \neq 4} (\langle A_\alpha X_i, \nabla_{JY_j} JY_k \rangle Z_\alpha + \langle A_{\alpha*} X_i, \nabla_{JY_j} JY_k \rangle JZ_\alpha). \end{aligned} \quad (3.49)$$

Since the equation of Codazzi gives

$$(\bar{\nabla}_{X_i} \sigma)(JY_j, JY_k) = (\bar{\nabla}_{JY_j} \sigma)(X_i, JY_k), \quad (3.50)$$

the Z_1 -components of both sides of (3.50) yield

$$2\langle JY_k, \nabla_{X_1} Y_j \rangle = \delta_{jk} \theta_{21}(X_1), \quad (3.51)$$

where we used (3.39), (3.40) and the fact that X_i and Y_k are orthogonal. Similarly, by comparing the JZ_1 -, JZ_2 -, and JZ_3 -components of (3.50), we may also obtain

$$2\langle JY_k, \nabla_{Y_j} X_i \rangle = \delta_{ik} \theta_{13}(JY_j) + \sum_{\alpha \geq 4} \langle A_\alpha X_i, Y_k \rangle \theta_{1\alpha}(JY_j), \quad (3.52)$$

$$2\langle Y_k, \nabla_{Y_j} X_i \rangle = \delta_{ik} \theta_{23}(JY_j) + \sum_{\alpha \geq 4} \langle A_\alpha X_i, Y_k \rangle \theta_{2\alpha}(JY_j), \quad (3.53)$$

$$\begin{aligned} & -\delta_{jk} \theta_{13}(X_i) + \langle JX_k, \nabla_{X_1} Y_j \rangle + \langle JX_j, \nabla_{X_1} Y_k \rangle \\ & = \sum_{\alpha \geq 4} \langle A_\alpha X_i, Y_k \rangle \theta_{\alpha 3}(JY_j) - \langle X_k, \nabla_{Y_j} X_i \rangle - \langle Y_i, \nabla_{Y_j} Y_k \rangle, \end{aligned} \quad (3.54)$$

where we used (3.51) to derive (3.53).

Since $A_\alpha A_3 + A_3 A_\alpha = 0$ for $\alpha \geq 4$, Lemma 5 implies

$$\langle A_\alpha X_i, Y_k \rangle = -\langle A_\alpha X_k, Y_i \rangle. \quad (3.55)$$

Therefore, (3.52) and (3.53) yield

$$\delta_{ik} \theta_{13}(JY_j) = \langle JY_k, \nabla_{Y_j} X_i \rangle + \langle JY_i, \nabla_{Y_j} X_k \rangle, \quad (3.56)$$

$$\delta_{ik} \theta_{23}(JY_j) = \langle Y_k, \nabla_{Y_j} X_i \rangle + \langle Y_i, \nabla_{Y_j} X_k \rangle. \quad (3.57)$$

Furthermore, from (3.55), we see that the left-hand side of (3.54) is symmetric with respect to the indices j and k and the right-hand side is skew-symmetric with respect to j and k , thus we obtain

$$\delta_{jk} \theta_{13}(X_i) = \langle JY_j, \nabla_{X_1} X_k \rangle + \langle JY_k, \nabla_{X_1} X_j \rangle, \quad (3.58)$$

$$\langle \nabla_{Y_j} Y_i, Y_k \rangle - \langle \nabla_{Y_j} X_i, X_k \rangle = \sum_{\alpha \geq 4} \langle A_\alpha X_i, Y_k \rangle \theta_{3\alpha}(JY_j). \quad (3.59)$$

From (3.51) (respectively, (3.52), (3.53) and (3.59)), we obtain (3.42) for T in V' (respectively, (3.43), (3.44), and (3.45) for T in JV''). By using the same method, we may obtain (3.41) - (3.45) for all T in TN . (The computation is long, but straight-forward). (Q.E.D.)

In the following, we denote by R and $R^\perp$ the Riemann curvature tensor and the normal curvature tensor of the leaf N .

LEMMA 7. *Under the hypothesis and the notations of Lemma 5, we have*

$$\begin{aligned} & 2R(X_1, Y_1; Y_1, X_1) + 2\langle \nabla_{Y_1} Y_1, \nabla_{X_1} X_1 \rangle - 2\langle \nabla_{X_1} Y_1, \nabla_{Y_1} X_1 \rangle \\ & = R^\perp(X_1, Y_1; Z_3, Z_2) + \langle D_{Y_1} Z_3, D_{X_1} Z_2 \rangle - \langle D_{X_1} Z_3, D_{Y_1} Z_2 \rangle. \end{aligned} \quad (3.60)$$

PROOF. From Lemma 5, we have $\langle A_\alpha X_1, Y_1 \rangle = \langle A_\alpha X_1, A_3 X_1 \rangle = \langle X_1, A_\alpha A_3 A_1 \rangle = 0$ for $\alpha \geq 4$. Thus Lemma 6 implies $2\langle \nabla_T Y_1, X_1 \rangle = \theta_{3,2}(T) = \langle D_T Z_3, Z_2 \rangle$, from which we obtain (3.60). (Q.E.D.)

4. PROOF OF THEOREM 1.

Under the hypothesis of Theorem 1, if M is non-proper, Lemma 3 implies $p = \dim_{\mathbb{R}} \mathcal{H}^\perp \geq 3$.

If $p \geq 4$, then Lemmas 5 and 6 imply that, for $i \geq 2$, we have

$$\begin{aligned}
2\langle \nabla_T Y_1, X_i \rangle &= \sum_{\alpha \geq 4} \langle A_\alpha X_i, Y_1 \rangle \theta_{\alpha 2}(T) \\
&= \sum_{\alpha \geq 4} \langle A_\alpha X_1, X_i \rangle \theta_{\alpha 2}(T) = \sum_{\alpha \geq 4} \langle X_{\alpha-2}, X_i \rangle \theta_{\alpha 2}(T).
\end{aligned}$$

Thus, we have $2\langle \nabla_T Y_1, X_i \rangle = \theta_{i+2,2}(T)$, $i = 2, \dots, r$. Similarly, we have $2\langle \nabla_T X_1, Y_i \rangle = \theta_{i+2,2}(T)$, $i = 2, \dots, r$. Thus, by applying Lemma 6, we may obtain

$$\begin{aligned}
&2\langle \nabla_{Y_1} Y_1, \nabla_{X_1} X_1 \rangle - 2\langle \nabla_{X_1} Y_1, \nabla_{Y_1} X_1 \rangle \\
&= \sum_{i=2}^{p-2} \theta_{i+2,2}(Y_1) [\langle \nabla_{X_1} X_1, X_i \rangle - \langle \nabla_{X_1} Y_1, Y_i \rangle] \\
&\quad + \sum_{i=2}^{p-2} \theta_{i+2,2}(X_1) [\langle \nabla_{Y_1} Y_1, Y_i \rangle - \langle \nabla_{Y_1} X_1, X_i \rangle] \\
&\quad + 2\theta_{1,2}(X_1) \langle \nabla_{Y_1} X_1, JY_1 \rangle + 2\theta_{2,1}(Y_1) \langle \nabla_{X_1} X_1, JY_1 \rangle.
\end{aligned}$$

Therefore, by applying Lemma 6 again, we may find

$$2\langle \nabla_{Y_1} Y_1, \nabla_{X_1} X_1 \rangle - 2\langle \nabla_{X_1} Y_1, \nabla_{Y_1} X_1 \rangle = \langle D_{X_1} Z_2, D_{Y_1} Z_3 \rangle - \langle D_{Y_1} Z_2, D_{X_1} Z_3 \rangle. \quad (4.1)$$

Combining (4.1) with (3.60) of Lemma 7, we get

$$2R(X_1, Y_1; Y_1, X_1) = R^\perp(X_1, Y_1; Z_3, Z_2). \quad (4.2)$$

From (2.7), (3.46), (3.47), Lemma 5 and the equation of Gauss, we may find

$$R(X_1, Y_1; Y_1, X_1) = -2. \quad (4.3)$$

On the other hand, (2.7), the equation of Ricci, Lemma 1 and Lemma 5 give

$$R^\perp(X_1, Y_1; Z_3, Z_2) = 2\langle A_2 X_1, X_1 \rangle = 2. \quad (4.4)$$

Equations (4.2), (4.3) and (4.4) give a contradiction. If $p = 3$, then, by (3.27) and the equation of Codazzi, we may obtain (3.41) - (3.45) in such forms that the summation terms in (3.43) - (3.45) were disappeared. By applying these equations, we may obtain a contradiction in a similar way. (Q.E.D.)

REMARK. For a CR-submanifold M of a Kaehler manifold, the condition that M is mixed-foliate is equivalent to $AP = -PA$.

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