A NOTE ON THE VERTEX-SWITCHING RECONSTRUCTION

I. KRASIKOV
School of Mathematical Sciences
Tel-Aviv University
Tel-Aviv
Israel

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ABSTRACT. Bounds on the maximum and minimum degree of a graph establishing its
reconstructibility from the vertex switching are given. It is also shown that any
disconnected graph with at least five vertices is reconstructible.

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1. INTRODUCTION.

A switching $G_v$ of a graph $G$ at vertex $v$ is a graph obtained from $G$ by deleting
all edges incident to $v$ and inserting all possible edges to $v$ which are not in $G$.
Since switching is a commutative operation, i.e., $(G_v)_u = (G_u)_v$, the definition can
be naturally extended to arbitrary subsets of the vertex set $V(G)$. Thus, $G_{A}$ is defined
for all $A \subseteq V(G)$.

The Vertex-Switching Reconstruction Problem, proposed by Stanley [1], asks: Is
$G$ uniquely determined up to isomorphism by the set (deck $D$), $\{G_v\}_v \in V(G)$?
If the answer is "yes" then $G$ is called reconstructible.

It was shown in [1] that any graph $G$ with $n = |V(G)| \equiv 0(\text{mod } 4)$ is reconstructible.
It seems that a little is known about the case $n = 0(\text{mod } 4)$. However, Stanley
pointed out [1], that the degree sequence of a graph, and consequently, the number of
edges easily reconstructible, provided $n \neq 4$. Bounds on the number of edges in a
graph, $e(G)$, establishing its reconstructibility was given [2]. Namely:

$$e(G) \leq \left[\frac{n(n-2)}{4}, \frac{n^2}{4}\right], \quad n \neq 4.$$

As might be expected, in virtue of the last result, $G$ is reconstructible if it
has a vertex of degree not close to $n/2$ or if $G$ is disconnected. Here we will prove
the last claim (Theorem 2) and show that for sufficiently large $n$ a graph is recon-
structible if $\max (\Delta, n - \delta) > 0.9n$, where $\Delta$ and $\delta$ are the maximum and the minimum
degree of $G$ respectively. Actually, we prove a little more, namely:
2. MAIN RESULTS.

THEOREM 1. If \[ \min \left( n \binom{n-1}{\Delta}, n \binom{n-1}{\delta} \right) < 2^{n/2-3} \], then \( G \) is reconstructible.

PROOF. In virtue of the quoted result of Stanley, we may assume \( n \equiv 0 \pmod{4} \). We will consider a graph \( G \) as a spanning subgraph of a fixed copy of the complete graph \( K_n \). The switching equivalence class \( G^* \) of \( G \) is the set of all \( H \subset K_n \) isomorphic to \( G \) such that \( H = G_A \) for some switching \( A \subset V(G) \).

For each subgraph \( g \subset G \), let \( \mu(G^* \ni g) \) be the number of those elements of \( G^* \) which contain a fixed copy of \( g \).

First we show that \( G \) is reconstructible if

\[ \frac{\mu(G^* \ni g)}{s(g \rightarrow G)} \leq \frac{1}{2} \]  

where \( s(H \rightarrow F) \) is the number of the subgraphs of \( F \) isomorphic to \( H \).

Observe that

\[ |G|^s(g \rightarrow G) \leq \mu(G^* \ni g) s(g \rightarrow K_n) \]  

On the other hand, consider the set \( S = \{ A : G_A \in G^* \} \).

Observe that \( |S| = 2|G^*| \) since \( G_A \) and \( G_{-A} \) are identical. It is known that for a nonreconstructible graph \( |S| \geq \left( \begin{array}{c} \frac{n}{2} \end{array} \right) \) ([2], Corollary 2.4). Thus, if \( G \) is not reconstructible then

\[ 2|G^*| \geq \sum \left( \begin{array}{c} \frac{n}{2} \end{array} \right) \]  

Comparing (2.2) and (2.3), we get that (2.1) is enough for the reconstructibility of \( G \).

Now we will prove that disconnected graphs are reconstructible. First we need the following simple lemma:

LEMMA 1. Suppose that nonisomorphic graphs \( G \) and \( H \) have the same deck. Then for any \( v \in V(G) \) there is \( u \in V(G) \), \( v \neq u \), such that \( G_vu \neq H \).

PROOF. Since the decks of \( G \) and \( H \) are equal then there is a bijection \( \phi : V(G) \rightarrow V(H) \) such that \( G_v \neq H_{\phi(v)} \). Let \( h_v : H_{\phi(v)} \rightarrow G_v \) be an isomorphism. Choosing \( u = h(\phi(v)) \) we obtain \( G_{vu} \neq H \). Moreover, since \( G_{vv} = G \), then \( v \neq \phi(v) \).
COROLLARY 1. Let \( n \neq 4 \). If \( G_{vu} \) and \( G, \_ \neq u \), have the same deck then 
\[ \deg (v) + \deg (u) = n \text{ or } n - 2, \]
depending on whether \( v \) and \( u \) are adjacent in \( G \) or are not.

PROOF. Let \( e(v,u) \) be the number of edges between \( v \) and \( u \). Since \( e(G) = e(H) \) then
\[ \deg (v) + \deg (u) = 2e(v,u) = \frac{1}{2} \cdot 2(n - 2) = n - 2. \]

COROLLARY 2. If \( G \) is not reconstructible and \( n \neq 4 \) then \( n - 2 \leq \delta + \Delta \leq n \).

PROOF. This easily follows from Lemma 1 and Corollary 1. We omit the details.

THEOREM 2. Any disconnected graph is reconstructible, provided \( n \neq 4 \).

PROOF. Assume the contrary. Then there are two nonisomorphic graphs \( G \) and \( H \) with the same deck, \( n \neq 4 \), and, say, \( G \) is disconnected. Denote by \( C \) a minimal connected component of \( G \). First we show that \( G \) has exactly two connected components and \( C \neq K_{6+1} \).

Let \( v \) be a vertex of the minimal degree in \( C \), and let \( u \) be such a vertex that 
\[ G_{vu} \neq H. \]
We claim that either \( u \neq v \) or \( G \) is regular of degree \( n - 2 \). Indeed,
\[ C > \max (\deg (v) + 1, \deg (u) + 1) > n/2, \]
which contradicts the minimality of \( C \). Furthermore, if \( G \) is regular then again \( v \) and \( u \) are in different components since, otherwise, the degree sequences of \( G \) and \( G_{vu} \) are different. Now it follows by Corollary 1, \( \deg(v) + \deg (u) = n - 2 \). Therefore, \( G \) has exactly two components, \( C \) is regular, and \( \Delta \geq n/2. \)

Let us show that \( C \) is just \( K_{6+1} \). Since all vertices of degree \( \Delta \) are in \( C \), we have
\[ \deg (v) + 1 \leq |C| \leq n - \Delta - 1. \]

Hence, applying Corollary 2, we get
\[ n - 2 \leq \delta + \Delta \leq \deg (v) + \Delta \leq n - 2. \]

Thus, \( \deg (v) = \delta, \deg (u) = \Delta, \) and \( C \neq K_{6+1} \).

Finally, \( G_{vu} \neq G \) since \( \deg (v) = |C| - 1, u \in \_ \) and \( \deg (u) = \Delta = |\_| - 1, \)
which is a contradiction. This completes the proof.

REFERENCES
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