SOME APPLICATIONS OF SCHWARZ LEMMA FOR OPERATORS

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ABSTRACT. A generalized Schwarz lemma and some Harnack type inequalities for operators have been obtained in this paper.

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1. INTRODUCTION.

Let A be a bounded linear operator on a complex Hilbert space H . For a complex valued function f analytic on a domain E of the complex plane containing the spectrum $\sigma(A)$ of A , let $f(A)$ denote the operator on H defined by the Riesz Dunford integral ([2, p.568]).

$$f(A) = \frac{1}{2\pi i} \int_C f(z) (zI - A)^{-1} dz,$$

where C is a positively oriented simple closed rectifiable contour containing $\sigma(A)$ in its inside domain Ω and satisfying $C \cup \Omega \subset E$. Fan [3] has obtained Schwarz lemma for $f(A)$ and has given several applications of his results including the Harnack's inequalities for operators in [3,4].

In this paper, we obtain a generalized Schwarz lemma and some further Harnack type inequalities for operators.

2. SOME PRELIMINARY LEMMAS.

We need the following lemmas.

LEMMA 1. Let a, b, c, d be complex numbers such that $ad - bc \neq 0$, $c \neq 0$ and let T be a bounded linear operator on a Hilbert space H such that $-d/c$ is not in $\sigma(T)$. Then

$$\| (aT + bI) (cT + dI)^{-1} \| \leq r \quad (2.1)$$

for $0 < r < |a|/|c|$ if and only if

$$\| T + \frac{\bar{a}b - r^2 \bar{c}d}{|a|^2 - r^2 |c|^2} I \| \leq \frac{r|ad - bc|}{|a|^2 - r^2 |c|^2} \quad (2.2)$$

Equality holds in (2.1) and (2.2) simultaneously.

PROOF. The inequality (2.1) is true if and only if

$$r^2 I - (\bar{c}T^* + \bar{d}I)^{-1} (\bar{a}T^* + \bar{b}I) (aT + bI) (cT + dI)^{-1} \geq 0$$

or

$$(\bar{c}T^* + \bar{d}I)^{-1} [r^2 (\bar{c}T^* + \bar{d}I) (cT + dI) - (\bar{a}T^* \bar{b}I)(aT + bI)] (cT + dI)^{-1} \geq 0$$

The operator inside the square brackets can be written as

$$\frac{r^2 |ad - bc|}{|a|^2 - r^2 |c|^2} I - \left\{ T^* T + \frac{\bar{a}b - r^2 \bar{c}d}{|a|^2 - r^2 |c|^2} T^* + \frac{\bar{a}b - r^2 \bar{c}d}{|a|^2 - r^2 |c|^2} T + \frac{|\bar{a}b - r^2 \bar{c}d|^2}{(|a|^2 - r^2 |c|^2)^2} I \right\}$$

or

$$\frac{r^2 |ad - bc|}{|a|^2 - r^2 |c|^2} I - \left[T^* + \frac{\bar{a}b - r^2 \bar{c}d}{|a|^2 - r^2 |c|^2} I \right] \left[T + \frac{\bar{a}b - r^2 \bar{c}d}{|a|^2 - r^2 |c|^2} I \right]$$

This last expression is a positive operator if and only if (2.2) holds. This completes the proof.

LEMMA 2. Let a, b, c, d and T be as in Lemma 1. Then,

$$\|(aT + bI)(cT + dI)^{-1} - \frac{\bar{b}d - r^2 \bar{a}c}{|d|^2 - r^2 |c|^2} I\| \leq \frac{r|ad - bc|}{|d|^2 - r^2 |c|^2} \quad (2.3)$$

for $0 < r < |d| / |c|$ if and only if $\|T\| \leq r$. Equality holds in (2.3) if and only if $\|T\| = r$.

PROOF. The inequality (2.3) is equivalent to

$$\| \{ (aT + bI) (|d|^2 - r^2 |c|^2)^{-1} - (\bar{b}d - r^2 \bar{a}c) (cT + dI) \} (cT + dI)^{-1} \| \leq r |ad - bc|$$

After simplification the above can be written as

$$\| (\bar{d}T + r^2 \bar{c}I) (cT + dI)^{-1} \| \leq r \quad (2.4)$$

Now an application of Lemma 1 shows that (2.4) is equivalent to $\|T\| \leq r$.

3. A GENERALIZED SCHWARZ LEMMA.

Let D denote the open unit disc $\{z: |z| < 1\}$ in the complex plane and let $H(D)$ be the class of complex valued functions analytic in D . Further, let $B(D) = \{f \in H(D): |f(z)| < 1, z \in D\}$ and let $B_0(D) = \{f \in B(D): f(0) = 0\}$.

THEOREM 1. Let f be in $B(0)$ and let A be a proper contraction on a Hilbert space H . Then,

$$\frac{\|A\| - |f(0)|}{1 - |f(0)| \|A\|} \leq \|f(A)\| \leq \frac{\|A\| + |f(0)|}{1 + |f(0)| \|A\|}. \quad (3.1)$$

PROOF. Since f is in $B(D)$ and A is a proper contraction, by a result of F and ([3, Theorem 1, p.276]), $T = f(A)$ is also a proper contraction. Now, if we define the complex valued function g by $g(z) = (f(z) - f(0)) (1 - \overline{f(0)} f(z))^{-1}$ then g is in $B_0(D)$ and $g(A) = (T - f(0)I) (I - \overline{f(0)} T)^{-1}$ is also a proper contraction. Further, by the operator version of Schwarz lemma ([3, Corollary 2, p.280]),

$$\|g(A)\| \leq \|A\|. \quad (3.2)$$

If we take $a = d = 1$, $b = f(0)$, $c = -\overline{f(0)}$ and $r = \|A\|$ in Lemma 1 then (3.2) is equivalent to

$$\|f(A) - \frac{1 - \|A\|^2 f(0)}{1 - |f(0)|^2 \|A\|^2} I\| \leq \frac{(1 - |f(0)|^2) \|A\|}{1 - |f(0)|^2 \|A\|^2}.$$

Using triangle inequality we get both the inequalities in (3.1).

COROLLARY 1. Let f in $B_0(D)$ be given by the series $f(z) = bz^n + \dots$, $b \neq 0$, and let A be a proper contraction on a Hilbert space H . Then

$$\|A^n\| \left(\frac{\|A\| - |b|}{1 - |b| \|A\|} \right) \leq \|f(A)\| \leq \|A^n\| \left(\frac{\|A\| + |b|}{1 + |b| \|A\|} \right). \quad (3.3)$$

PROOF. The function g , defined by $g(z) = (f(z)/z^n)$, $z \neq 0$ and $g(0)=b$, is in $B(D)$ and $f(A) = A^n g(A)$. Hence the result follows from Theorem 1.

REMARK. The author learned from Professor R. Finn that Theorem 1 follows independently from some results of K. Fan that are now in press.

4. SOME HARNACK TYPE INEQUALITIES.

Let $P(\alpha, \beta)$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, denote the subclass of functions p in $H(D)$ satisfying $p(0) = 1$ and

$$\frac{|p(z) - 1|}{|(2\beta - 1)p(z) + (1 - 2\alpha\beta)|} < 1, \quad z \text{ in } D.$$

This class of functions have been introduced and studied by Juneja and Mogra [5].

It has been shown in [5] that the n^{th} Taylor coefficient a_n of a function p in $P(\alpha, \beta)$ satisfies the sharp inequality $|a_n| \leq 2\beta(1-\alpha)$. Observe that

$$\begin{aligned} P(0,1) &= \{p \in H(D): p(0)=1, \operatorname{Re} p(z) > 0, z \text{ in } D\}, \\ P(\alpha,1) &= \{p \in H(D): p(0)=1, \operatorname{Re} p(z) > \alpha, z \text{ in } D\}, \\ P(0,\beta) &= \{p \in H(D): p(0)=1, |p(z) - \frac{1}{2(1-\beta)}| < \frac{1}{2(1-\beta)}\}, \end{aligned}$$

and

$P(\alpha, \beta) \subset P(0, 1)$, for all admissible choices of α and β . We prove the following theorem which extends a distortion theorem by Kapoor and the author ([6, Theorem 1, p.86]).

THEOREM 2. Let p in $P(\alpha, \beta)$ be given by the series $p(z) = 1 + 2b(1-\alpha)\beta z^a + \dots$, $0 < |b| \leq 1$, z in D and let A be a proper contraction on a Hilbert space H . Then,

$$\|p(A)\| \leq \frac{1 + \|A\| |b| + (1-2\alpha\beta)(\|A\| + |b|)\|A^n\|}{1 + \|A\| |b| + (1-2\beta)(\|A\| + |b|)\|A^n\|}, \quad (4.1)$$

$$\|P(A)\| \geq \frac{1 + \|A\| |b| - (1-2\alpha\beta)(\|A\| + |b|)\|A^n\|}{1 + \|A\| |b| - (1-2\beta)(\|A\| + |b|)\|A^n\|}, \quad (4.2)$$

$$\frac{1 + \|A\| |b| - (1-2\alpha\beta)(\|A\| + |b|)\|A^n\|}{1 + \|A\| |b| - (1-2\beta)(\|A\| + |b|)\|A^n\|} I \leq \operatorname{Re} p(A), \quad (4.3)$$

$$\operatorname{Re} p(A) \leq \frac{1 + \|A\| |b| + (1-2\alpha\beta)(\|A\| + |b|)\|A^n\|}{1 + \|A\| |b| + (1-2\beta)(\|A\| + |b|)\|A^n\|} I, \quad (4.4)$$

$$\pm \operatorname{Im} p(A) \leq \frac{2\beta(1-\alpha)(\|A\| + |b|)(1 + \|A\| |b|)\|A^n\|}{(1 + \|A\| |b|)^2 - (1-2\beta)^2(\|A\| + |b|)\|A^n\|^2} I. \quad (4.5)$$

PROOF. From the definition of $P(\alpha, \beta)$, it follows that there exists a function w in $B_0(D)$ such that

$$p(z) = \{1 + (1-2\alpha\beta)w(z)\} \{1 + (1-2\beta)w(z)\}^{-1}, \quad z \text{ in } D$$

and

$$p(A) = \{1 + (1-2\alpha\beta)T\} \{1 + (1-2\beta)T\}^{-1}, \quad \text{where } T = w(A).$$

Further, it is observed that $w(z) = bz^n + \dots$, where z in D . Hence by Corollary 1., we can say

$$\|T\| = \|w(A)\| \leq \|A^n\| \left(\frac{\|A\| + |b|}{1 + \|A\| |b|} \right) = r. \quad (4.6)$$

Now, choosing $a = 1-2\alpha\beta$, $c = 1-2\beta$, $b = d-1$ in Lemma 2, (4.6) is equivalent to

$$\|p(A) - \frac{1-r^2(1-2\alpha\beta)(1-2\beta)}{1-r^2(1-2\beta)^2}\| \leq \frac{2r(1-\alpha)\beta}{1-r^2(1-2\beta)^2}.$$

Hence

$$\frac{1-(1-2\alpha\beta)r}{1-(1-2\beta)r} \leq \|p(A)\| \leq \frac{2+(1-2\alpha\beta)r}{1-(1-2\beta)r}.$$

Substituting the value of r in the above inequality, we get (4.1) and (4.2).

Also,

$$\begin{aligned} \pm \operatorname{Re} [p(A) - \frac{1-r^2(1-2\alpha\beta)(1-2\beta)}{1-(1-2\beta)^2} I] &\leq \| p(A) - \frac{1-r^2(1-2\alpha\beta)(1-2\beta)}{1-r^2(1-2\beta)^2} I \| \\ &\leq \frac{2r(1-\alpha)^\beta}{1-r^2(1-2\beta)^2} I. \end{aligned}$$

This gives (4.3) and (4.4). Similarly,

$$\begin{aligned} \pm \operatorname{Im} p(A) &= \pm \operatorname{Im} [p(A) - \frac{1-r^2(1-2\alpha\beta)(1-2\beta)}{1-r^2(1-2\beta)^2} I] \\ &\leq \| p(A) - \frac{1-r^2(1-2\alpha\beta)(1-2\beta)}{1-r^2(1-2\beta)^2} I \| \leq \frac{2\beta(1-\alpha)r}{1-r^2(1-2\beta)^2} I. \end{aligned}$$

From this (4.5) follows. This completes the proof.

REMARK. The right hand side of (4.1) and (4.2) are increasing and decreasing function of $|b|$, respectively. For the case $|b|=1$, $\alpha=0$ and $\beta=1$, our Theorem 2 includes some results of Fan ([3, Corollary 3, P.281], [4, Proposition 2, P.335]).

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