ON (J, p_n) SUMMABILITY OF FOURIER SERIES

S.M. MAZHAR

Department of Mathematics
Kuwait University
P.O. BOX 5969
13060, Safat, Kuwait

(Received December 13, 1986 and in revised form April 8, 1987)

ABSTRACT. In this note two theorems have been established. The first one deals with the summability (J, p_n) of a Fourier series while the second one concerns with the summability of the first derived Fourier series. These results include, as a special case, certain results of Nanda [1].

KEY WORDS AND PHRASES. (J, p_n) summability, Fourier series, Derived Fourier series.
1980 AMS SUBJECT CLASSIFICATION CODES. 42A24, 40G10.

INTRODUCTION.

Let $\{p_n\}$ be a sequence of non-negative numbers such that $\sum_{n=0}^{\infty} p_n$ diverges and let the radius of convergence of power series

$$p(z) = \sum_{n=0}^{\infty} p_n x^n \quad (1.1)$$

be 1. Given any series $\sum a_n$ with the sequence of partial sums $\{s_n\}$ we write

$$p_s(x) = \sum_{n=0}^{\infty} p_n s_n x^n \quad (1.2)$$

and

$$J_s(x) = \frac{p_s(x)}{p(x)}. \quad (1.3)$$

If the series in (1.2) is convergent in $[0, 1)$ and

$$\lim_{x \rightarrow 1^-} J_s(x) = s,$$

we say that the series $\sum a_n$ or the sequence $\{s_n\}$ is summable (J, p_n) to s , where s is a finite number. ([2], [3], p.80).

For $p_n=1, \frac{1}{n}$ with $p_0=0$, and $A_n^k, k > -1$ we get summability A , summability (L) and A_k method of summability respectively.

Suppose f is a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let

$$f(x) \sim \frac{a_0}{2} + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_0^{\infty} A_n(x). \quad (1.4)$$

Then the first derived series of (1.4) is

$$\sum_1^{\infty} n(b_n \cos nx - a_n \sin nx) = \sum_1^{\infty} n B_n(x). \quad (1.5)$$

We write

$$\phi(t) = \frac{1}{2} \{f(x_0 + t) + f(x_0 - t) - 2s\}$$

$$\psi(t) = \frac{1}{2} \{f(x_0 + t) - f(x_0 - t)\}$$

$$g(t) = \frac{\psi(t)}{2s \sin t/2} - s$$

$$M(t) = \sum_0^{\infty} p_n x^n \sin nt$$

$$\Phi(t) = \int_t^{\pi} \frac{\phi(u)}{u} du$$

and

$$G(t) = \int_t^{\pi} \frac{g(u)}{u} du.$$

2. MAIN RESULTS.

In this note we propose to establish the following theorems on (J, p_n) summability of (1.4) and (1.5).

THEOREM 1. Let $\{p_n\}$ be a positive sequence such that

$$(a) \quad n p_n = O(1), \quad \sum_{k=v}^{\infty} \frac{p_k}{k+1} = O(p_v) \text{ and}$$

$$(b) \quad \sum_{n=0}^{\infty} |\Delta^2(n p_n x^n)| = O(1-x), \quad 0 < x < 1.$$

If $\phi(t) = o(p(1-t))$, $t \rightarrow +0$, then the Fourier series (1.4) is summable (J, p_n) to s .

THEOREM 2. Let $\{p_n\}$ satisfy the hypothesis (a) of Theorem 1. If

$$(c) \quad \sum_0^{\infty} n |\Delta^3(n p_n x^n)| = O(1-x), \quad 0 < x < 1$$

and $G(t) = o(p(1-t))$, as $t \rightarrow 0+$, then the first derived series (1.5) is summable (J, p_n) to s .

It may be remarked that for $p_n = \frac{1}{n}$, $n \geq 1$, $p_0 = 0$ our theorems include two known theorems of Nanda [1] on L -summability of Fourier series and its derived series. For an earlier result on (J, p_n) summability of (1.4) under more stringent conditions see

Khan [4]. Very recently in 1985 Prem Chandra, Mohapatra and Sahney [5] have established a similar theorem on (J, p_n) summability with another set of conditions. It may be observed that $n p_n = O(1)$ $p'(x) = O(\frac{1}{1-x})$ as $x \rightarrow 1^-$. Also it is easy to see that $n p_n < (n+1) p_{n+1}$, $n = 0, 1, 2, \dots$ and $p'(x) = O(\frac{1}{1-x})$ imply that $n p_n = O(1)$. For if $\{n p_n\}$ is not bounded then $\lim_{n \rightarrow \infty} n p_n = \infty$. Now using the well known result that

$$\lim_{n \rightarrow \infty} \frac{\beta_n}{\alpha_n} = \infty \quad \lim_{x \rightarrow 1^-} \frac{\sum_{n=0}^{\infty} \beta_n x^n}{\sum_{n=0}^{\infty} \alpha_n x^n} = \infty,$$

where radius of convergence of each power series is 1 and $\alpha > 0$ with

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \text{ we find for } \alpha_n = 1, \beta_n = n p_n \text{ that}$$

$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} n p_n x^n = \infty$. This means $\lim_{x \rightarrow 1^-} (1-x)p'(x) = \infty$ which contradicts the hypothesis that $(1-x)p'(x) = O(1)$ as $x \rightarrow 1^-$. Thus conditions (2.1) and (2.4) of [5] imply that $n p_n = O(1)$.

3. PROOF OF THEOREM 1.

Let $s_n(x_0)$ denote the n -th partial sum of (1.4) at $x = x_0$. Then

$$s_n(x_0) - s = \frac{2}{\pi} \int_0^{\pi} \phi(t) \frac{\sin nt}{t} dt + o(1)$$

so that

$$\begin{aligned} \sum_{n=0}^{\infty} p_n x^n (s_n(x_0) - s) &= \frac{2}{\pi} \int_0^{\pi} \frac{\phi(t)}{t} \sum_{n=0}^{\infty} p_n x^n \sin nt dt + o(p(x)) \\ &= \frac{2}{\pi} \int_0^{\pi} -\phi'(t) M(t) dt + o(p(x)) \\ &= \frac{2}{\pi} \int_0^{\pi} \phi(t) \sum_{n=0}^{\infty} n p_n x^n \cos nt dt + o(p(x)) \\ &= \frac{2}{\pi} \left(\int_0^{1-x} + \int_{1-x}^{\pi} \right) \dots + o(p(x)) \\ &= I_1 + I_2 + o(p(x)), \text{ say.} \end{aligned} \tag{3.1}$$

Now

$$\begin{aligned} I_1 &= \frac{2}{\pi} \int_0^{1-x} \phi(t) \sum_{n=0}^{\infty} n p_n x^n \cos nt dt \\ &= o\left(\int_0^{1-x} o(p(1-t)) \frac{1}{1-x} dt\right) \\ &= o\left(\frac{1}{1-x}\right) \int_0^{1-x} \sum_{n=0}^{\infty} p_k (1-t)^k dt \end{aligned} \tag{3.2}$$

Khan [4]. Very recently in 1985 Prem Chandra, Mohapatra and Sahney [5] have established a similar theorem on (J, p_n) summability with another set of conditions. It may be observed that $n p_n = O(1)$ $p'(x) = O(\frac{1}{1-x})$ as $x \rightarrow 1^-$. Also it is easy to see that $n p_n < (n+1) p_{n+1}$, $n = 0, 1, 2, \dots$ and $p'(x) = O(\frac{1}{1-x})$ imply that $n p_n = O(1)$. For if $\{n p_n\}$ is not bounded then $\lim_{n \rightarrow \infty} n p_n = \infty$. Now using the well known result that

$$\lim_{n \rightarrow \infty} \frac{\beta_n}{\alpha_n} = \infty \quad \lim_{x \rightarrow 1^-} \frac{\sum_{n=0}^{\infty} \beta_n x^n}{\sum_{n=0}^{\infty} \alpha_n x^n} = \infty,$$

where radius of convergence of each power series is 1 and $\alpha > 0$ with

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \text{ we find for } \alpha_n = 1, \beta_n = n p_n \text{ that}$$

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} n p_n x^n = \infty. \text{ This means } \lim_{x \rightarrow 1^-} (1-x)p'(x) = \infty \text{ which contradicts the}$$

hypothesis that $(1-x)p'(x) = O(1)$ as $x \rightarrow 1^-$. Thus conditions (2.1) and (2.4) of [5] imply that $n p_n = O(1)$.

3. PROOF OF THEOREM 1.

Let $s_n(x_0)$ denote the n -th partial sum of (1.4) at $x = x_0$. Then

$$s_n(x_0) - s = \frac{2}{\pi} \int_0^{\pi} \phi(t) \frac{\sin nt}{t} dt + o(1)$$

so that

$$\begin{aligned} \sum_{n=0}^{\infty} p_n x^n (s_n(x_0) - s) &= \frac{2}{\pi} \int_0^{\pi} \frac{\phi(t)}{t} \sum_{n=0}^{\infty} p_n x^n \sin nt dt + o(p(x)) \\ &= \frac{2}{\pi} \int_0^{\pi} -\phi'(t) M(t) dt + o(p(x)) \\ &= \frac{2}{\pi} \int_0^{\pi} \phi(t) \sum_{n=0}^{\infty} n p_n x^n \cos nt dt + o(p(x)) \\ &= \frac{2}{\pi} \left(\int_0^{1-x} + \int_{1-x}^{\pi} \right) \dots + o(p(x)) \\ &= I_1 + I_2 + o(p(x)), \text{ say.} \end{aligned} \tag{3.1}$$

Now

$$\begin{aligned} I_1 &= \frac{2}{\pi} \int_0^{1-x} \phi(t) \sum_{n=0}^{\infty} n p_n x^n \cos nt dt \\ &= o \left(\int_0^{1-x} o(p(1-t)) \frac{1}{1-x} dt \right) \\ &= o \left(\frac{1}{1-x} \right) \int_0^{1-x} \sum_{n=0}^{\infty} p_k (1-t)^k dt \end{aligned} \tag{3.2}$$

$$\begin{aligned}
&= \frac{o(1)}{1-x} \sum_{k=0}^{\infty} \frac{p_k}{k+1} (1-x^{k+1}) = o(1) \sum_{k=0}^{\infty} \frac{p_k}{k+1} \sum_{v=0}^k x^v \\
&= o(1) \sum_{v=0}^{\infty} x^v \sum_{k=v}^{\infty} \frac{p_k}{k+1} = o(1) \sum_{v=0}^{\infty} p_v x^v = o(p(x)).
\end{aligned}$$

Thus

$$I_1 = o(p(x)). \quad (3.3)$$

Again

$$\begin{aligned}
I_2 &= \frac{2}{\pi} \int_{1-x}^{\pi} \phi(t) \sum_{k=0}^{\infty} k p_k x^k \cos kt \\
&= \frac{2}{\pi} \int_{1-x}^{\pi} \phi(t) \sum_{k=0}^{\infty} \Delta^2 (k p_k x^k) F_k(t)
\end{aligned}$$

where

$$F_k(t) = \sum_{v=0}^k D_v(t) = \sum_{v=0}^k D_v(t) = \sum_{v=0}^k \frac{\sin(v+\frac{1}{2})t}{2 \sin t/2}.$$

Under the hypothesis of Theorem 1

$$\begin{aligned}
I_2 &= O(1) \int_{1-x}^{\pi} p(1-t) \frac{(1-x)}{t^2} dt \\
&= o(p(x)) \int_{1-x}^{\pi} \frac{(1-x)}{t^2} dt \\
&= o(p(x)). \quad (3.4)
\end{aligned}$$

Thus in view of (3.1), (3.3) and (3.4)

$$\sum_0^{\infty} p_n x^n (s_n(x_0) - s) = o(p(x)) \text{ as } x \rightarrow 1-.$$

This proves Theorem 1.

4. PROOF OF THEOREM 2. As shown in ([6], p. 54) we can assume that $s = 0$.

Let $T_n(x_0)$ denote the n -th partial sum of (1.5) at $x = x_0$. Then

$$\begin{aligned}
T_n(x_0) &= \frac{1}{\pi} \int_0^{\pi} \frac{g(t) \sin nt}{\sin t/2} dt - \frac{2n}{\pi} \int_0^{\pi} \cos(n + \frac{1}{2})t g(t) dt \\
&= \frac{2}{\pi} \int_0^{\pi} \frac{g(t) \sin nt}{t} dt + o(1) - \frac{2n}{\pi} \int_0^{\pi} \cos(n + \frac{1}{2})t g(t) dt, \\
&= T_{n,1} + o(1) + T_{n,2}, \text{ say,}
\end{aligned}$$

so that

$$\begin{aligned}
\sum_0^{\infty} T_n(x_0) p_n x^n &= \sum_0^{\infty} T_{n,1} p_n x^n + o(p(x)) + \sum_0^{\infty} T_{n,2} p_n x^n \\
&= L_1 + o(p(x)) + L_2, \text{ say.} \quad (4.1)
\end{aligned}$$

As shown in the proof of Theorem 1 in view of (4.5), $L_1 = o(p(x))$. (4.2)

$$\begin{aligned}
 \text{Let } H(t) &= \sum_0^\infty n p_n x^n \cos(n + \frac{1}{2})t. \text{ Then} \\
 L_2 &= -\frac{2}{\pi} \sum_0^\infty n p_n x^n \int_0^\pi \cos(n + \frac{1}{2})t g(t) dt \\
 &= -\frac{2}{\pi} \int_0^\pi g(t) \sum_{n=0}^\infty n p_n x^n \cos(n + \frac{1}{2})t dt \\
 &= -\frac{2}{\pi} \int_0^\pi g(t) H(t) dt = -\frac{2}{\pi} \int_0^\pi -t G'(t) H(t) dt \\
 &= \frac{2}{\pi} \{ [tG(t) H(t)]_0^\pi - \int_0^\pi G(t) \frac{d}{dt} (tH(t)) dt \} \\
 &= -\frac{2}{\pi} \int_0^\pi G(t) \{ H(t) + tH'(t) \} dt \\
 &= -\frac{2}{\pi} \left(\int_0^{1-x} + \int_{1-x}^\pi \right) = L_{21} + L_{22}, \text{ say.}
 \end{aligned}$$

Now since $n p_n = O(1)$

$$\begin{aligned}
 L_{21} &= \int_0^{1-x} \frac{o(p(1-t))}{1-x} dt \\
 &\quad + \int_0^{1-x} o(p(1-t)) t \sum_0^\infty n x^n dt \\
 &= o(p(x)) + o(1) \int_0^{1-x} \frac{p(1-t)}{1-x} dt = o(p(x))
 \end{aligned} \tag{4.3}$$

as shown in (3.2).

In view of the hypothesis of Theorem 2.

$$L_{22} = -\frac{2}{\pi} \int_{1-x}^\pi o(p(1-t)) \left| \frac{d}{dt} (t H(t)) \right| dt. \tag{4.4}$$

Since $n p_n = O(1)$, we have by using Abel's transformation

$$\begin{aligned}
 \frac{d}{dt} t H(t) &= \sum_0^\infty n p_n x^n \frac{d}{dt} t \cos(n + \frac{1}{2})t \\
 &= \sum_0^\infty \Delta(n p_n x^n) \frac{d}{dt} \frac{t}{2 \sin t/2} (\sin(n+1)t) \\
 &= \sum_0^\infty \Delta^2(n p_n x^n) \frac{d}{dt} \left\{ \frac{t}{4 \sin^2 t/2} (\cos t/2 - \cos(n+3/2)t) \right\}.
 \end{aligned}$$

Thus

$$\begin{aligned}
 \left| \frac{d}{dt} (t H(t)) \right| &\leq \frac{C}{t^2} \sum_{n=0}^\infty \left| \Delta^2 n p_n x^n \right| \\
 &\quad + \frac{C}{t} \left| \sum_0^\infty n \Delta^2(n p_n x^n) \sin(n+3/2)t \right| \\
 &\leq \frac{C(1-x)}{t^2} + \frac{C}{t} \left| \sum_0^\infty \Delta \{ n \Delta^2(n p_n x^n) \} \sum_{v=0}^n \sin(v+3/2)t \right|
 \end{aligned}$$

$$\leq \frac{C(1-x)}{t^2} + \frac{C}{t^2} \sum n |\Delta^3(n p_n x^n)|$$

$$\leq \frac{C(1-x)}{t^2},$$

where C is a positive constant not necessarily the same at each occurrence, and in view of the fact that $\sum_0^\infty n |\Delta^3(n p_n x^n)| = O(1-x)$ (4.5) implies that $\sum_0^\infty |\Delta^2(n p_n x^n)| = O(1-x)$. Hence from (4.4)

$$L_{22} = o(1) \int_{1-x}^1 p(1-t) \frac{(1-x)}{t^2} dt = o(p(x)), \text{ as shown in (3.3).} \quad (4.6)$$

Thus from (4.1) - (4.3) and (4.6) the proof of Theorem 2 follows.

REFERENCES

1. NANDA, M. The summability (L) of the Fourier series and the first differentiated Fourier series, Quart. J. Math. Oxford (2) 13 (1962), 229-234.
2. BORWEIN, D. On methods of summability based on power series, Proc. Royal Soc. Edinburgh 64 (1957), 342-349.
3. HARDY, G.H. Divergent Series (Oxford, 1949).
4. KHAN, F.M. On (J, p_n) summability of Fourier series, Proc. Edinburgh Math. Soc. 18 (II) (1972), 13-17.
5. CHANDRA, P., MOHAPATRA, R.N. and SAHNEY, B.N. Tamkang J. Math. 16 (1985) No. 2, 37-41.
6. MOHANTY, R. and NANDA, M. The summability by logarithmic means of the derived Fourier series, Quart. J. Math. Oxford (2) 6 (1955), 53-58.