COMPLETENESS OF REGULAR INDUCTIVE LIMITS

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ABSTRACT. Regular LB-space is fast complete but may not be quasi-complete. Regular inductive limit of a sequence of fast complete, resp. weakly quasi-complete, resp. reflexive Banach, spaces is fast complete, resp. weakly quasi-complete, resp. reflexive complete, space.

KEY WORDS AND PHRASES. Regular locally convex inductive limit, complete, quasi-complete, fast complete space.

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1. INTRODUCTION.

In [1, §31.6] Kothe has a sequence of Banach spaces $E_1 \subset E_2 \subset \dots$ whose inductive limit is not quasi-complete. In [2] there is an example of reflexive Frechet spaces E_n whose inductive limit is not even fast complete. Since an LF-space is fast complete iff it is regular, see [3], there is a natural question asked by Jorge Mujica in [4]: Is every regular LB-space complete?

Throughout the paper $E_1 \subset E_2 \subset \dots$ is a sequence of locally convex spaces with continuous inclusions $E_n \rightarrow E_{n+1}$, $n \in \mathbb{N}$. Their locally convex inductive limit is denoted by E . The space E is called regular if every set bounded in E is bounded in some E_n .

2. MAIN RESULTS.

Let F be a locally convex space and $A \subset F$ absolutely convex. We denote by F_A the seminormed space $U\{nA; n \in \mathbb{N}\}$ whose topology is generated by the Minkowski functional of A . If F_A is Banach space, A is called Banach disk. The space F is called fast complete if every set bounded in F is contained in a bounded Banach disk. Every sequentially complete space is fast complete and there are fast complete spaces which are sequentially incomplete, see [5].

EXAMPLE. For each $n \in \mathbb{N}$ and $x : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{C}$, put

$$\|x\|_n = \max \{ \sup \{ |x_{ij}|^{-i}; i \leq n, j \in \mathbb{N} \}, \sup \{ |x_{ij}|; i > n, j \in \mathbb{N} \} \},$$

$E_n = \{x; \|x\|_n < +\infty \text{ \& \; } \lim_{j \rightarrow \infty} x_{ij} = 0 \text{ for } i > n\}$, $B_n = \{x \in E_n; \|x\|_n \leq 1\}$, and

$E = \text{indlim } E_n$. We prove that each E_n is a Banach space, $E_1 \subset E_2 \subset \dots$, inclusions $E_n \rightarrow E_{n+1}$, $n \in \mathbb{N}$, are continuous, E is regular and not quasi-complete.

CLAIM 1. Each space E_n is Banach.

PROOF. Let $\{x(k)\}$ be a Cauchy sequence in E_n . For each $i, j \in \mathbb{N}$ the sequence $\{x(k)_{ij}\}$ is Cauchy in $\mathbb{C}$ and has a limit x_{ij} . Let x be the matrix with the entries x_{ij} .

Given $\epsilon > 0$, there is k such that $p, r \geq k$ implies $\|x(p) - x(r)\|_n \leq \epsilon$. Hence,

$$\|x(p) - x\|_n \leq \limsup_{r \rightarrow \infty} \|x(p) - x(r)\|_n \leq \epsilon \text{ and}$$

$$\|x\|_n \leq \|x - x(p)\|_n + \|x(p)\|_n < +\infty.$$

Take $i > n$ and choose j_i so that $|x(k)_{ij}| < \epsilon$ for $j > j_i$.

Then $|x_{ij}| \leq |x_{ij} - x(k)_{ij}| + |x(k)_{ij}| \leq \|x - x(k)\|_n + |x(k)_{ij}| < 2\epsilon$ and

$$\lim_{j \rightarrow \infty} x_{ij} = 0.$$

CLAIM 2. $E_1 \subset E_2 \subset \dots$ and each inclusion $E_n \rightarrow E_{n+1}$ is continuous. Proof follows from the inequalities $\|x\|_1 \geq \|x\|_2 \geq \dots$, $x \in \bigcup \{E_n; n \in \mathbb{N}\}$.

CLAIM 3. E is regular.

PROOF. Let $D \subset E$ be not bounded in any E_n . For each $n \in \mathbb{N}$ choose $x(n) \in D$ such that $\|x(n)\|_n > n$. There are $i(n), j(n) \in \mathbb{N}$ for which

$$|x(n)_{i(n),j(n)}| > \begin{cases} nj(n)^{i(n)} & \text{if } i(n) \leq n \\ n & \text{if } i(n) > n \end{cases}$$

Put $m(n) = n + \max \{i(k); k \leq n\}$ and $r(n) = \min \{j(k)^{-i(k)}; k \leq m(n)\}$, $n \in \mathbb{N}$.

If $k > n$ then $\|x(n)\|_{m(k)} \geq |_{m(k)} \geq |j(n)^{-i(n)} x(n)_{i(n),j(n)}| > nr(k)$,

if $k \leq n$ then $\|x(n)\|_{m(k)} \geq \|x(n)\|_{m(n)} > nr(k)$.

Let $V = \bigcup \{r(k)B_{m(k)}; k \in \mathbb{N}\}$ and $U = \text{co}V$. Assume $x(n) \in nU$. Then

$$x(n) = \sum_{k=1}^s \alpha_k y(k),$$

where $\alpha_k \geq 0$, $\sum \alpha_k = 1$, and $y(k) \in nr(k)B_{m(k)}$. To prove that $|y(k)_{i(n),j(n)}| \leq n$

for $k \in \mathbb{N}$, we have to distinguish three cases:

(a) $k > n$: Then $|y(k)_{i(n),j(n)}| = |y(k)_{i(n),j(n)} j(n)^{i(n)-i(n)}| \leq$

$$\leq \|y(k)\|_{m(k)} j(n)^{i(n)} \leq nr(k)j(n)^{i(n)} \leq n.$$

(b) $k \leq n$ & $i(n) \leq m(k)$: Then $|y(k)_{i(n),j(n)}| \leq \|y(k)\|_{m(k)} j(n)^{i(n)} \leq nr(k)j(n)^{i(n)} \leq n$.

(c) $k \leq n$ & $i(n) > m(k)$: Then $|y(k)_{i(n),j(n)}| \leq \|y(k)\|_{m(k)} \leq nr(k) \leq n$.

On the other hand $|x(n)_{i(n),j(n)}| > n$ and $x(n)$ cannot be a convex combination of $y(k)$, $k \leq s$, i.e. $x(n) \notin nU$. Since U is a 0-neighborhood in E , D is not bounded in E .

CLAIM 4. E is not quasi-complete.

PROOF. Let $\Delta = \{\delta \subset N \times N; \{j \in N; (i, j) \in \delta\} \text{ is finite, } i \in N\}$ be ordered by set inclusion. Denote by $x(\delta)$ the set characteristic function of $\delta \in \Delta$. Then $\{x(\delta); \delta \in \Delta\} \subset B_1$ and the filter associated with $\delta \rightarrow x(\delta)$ is bounded in E_1 , hence also bounded in E .

Let $P_n: C^{N \times N} \rightarrow C^{N \times N}$ be the projection of an $N \times N$ matrix on its n -th row. Take a closed absolutely convex 0-neighborhood V in E . For each $n \in N$ choose $m(n) \in N$ and $r(n) > 0$ so that $r(n)B_n \subset V$, $m(n) \geq 2r(n)^{-1/n}$, and put $\sigma = \{(i, j) \in N \times N; j \leq m(i)\}$.

If $\gamma, \delta \in \Delta$, $\gamma, \delta \supseteq \sigma$, then $x(\gamma)_{ij} - x(\delta)_{ij} = 0$ for $j \leq m(i)$ and $\|P_n(x(\gamma) - x(\delta))\|_n = \sup \{j^{-n} | x(\gamma)_{nj} - x(\delta)_{nj} |; j > m(n)\} < m(n)^{-n} \leq 2^{-n}r(n)$. Hence $2^n P_n(x(\gamma) - x(\delta)) \in r(n)B_n \subset V$. Since V is absolutely convex, the sequence

$$y_k = \sum_{n=1}^k 2^{-n} 2^{+n} P_n(x(\gamma) - x(\delta)), k \in N$$

is contained in V . It is also contained in B_1 and converges coordinate-wise to $x(\gamma) - x(\delta)$ in E_1 . Hence $x(\gamma) - x(\delta)$ is in the weak closure of V . Since V is closed and convex, it is also weakly closed and $x(\gamma) - x(\delta) \in V$. So $\{x(\delta); \delta \in \Delta\}$ is a base of a bounded Cauchy filter in E . If it had a limit $x \in E$, then $x_{ij} = 1$ for all $i, j \in N$. This would imply $x \notin E_n$ for any $n \in N$ and $x \notin E$, q.e.d.

LEMMA. Regular inductive limit of a sequence of semireflexive, resp. reflexive, spaces is semireflexive, resp. reflexive.

PROOF. Let each E_n be semireflexive. Since $E = \text{indlim } E_n$ is regular, its strong dual E'_b equals to $\text{projlim } (E_n)_b'$ and $(E'_b)' \subset \cup \{((E_n)_b')'; n \in N\} = \cup \{E_n; n \in N\} = E$.

Let each E_n be reflexive. By [7; IV, 5.6] it suffices to show that E is semireflexive and barreled. Take a barrel B in E . For each $n \in N$, $B \cap E_n$ is a barrel in E_n . Since E_n is reflexive, the barrel $B \cap E_n$ is a neighborhood in E_n , which implies that B is a neighborhood in E and E is barreled.

CONSEQUENCE. Inductive limit of a sequence of reflexive Banach spaces is reflexive.

PROOF. By [6; Th. 4] the inductive limit of reflexive Banach spaces is regular.

THEOREM. Let $E = \text{indlim } E_n$ be regular. Then:

- (a) Each E_n fast complete $\Rightarrow E$ fast complete.
- (b) Each E_n weakly quasi-complete $\Rightarrow E$ weakly quasi-complete.
- (c) Each E_n semireflexive $\Rightarrow E$ quasi-complete.
- (d) Each E_n reflexive Banach $\Rightarrow E$ complete.

PROOF.

- (a) Let $B \subset E$ be bounded, then it is bounded in some E_n and contained in a bounded Banach disk in E_n . Since any Banach disk bounded in E_n is also bounded in E , the proof is complete.
- (b) Follows from Lemma since any locally convex space is weakly quasi-complete iff it is semireflexive, [7;IV, 5.5].
- (c) Follows from (b) since every weakly quasi-complete space is quasi-complete.
- (d) Let $\mathcal{F}$ be a Cauchy filter in E . Then $\mathcal{F}$, as a filter of continuous linear functionals on E'_b , converges uniformly on bounded sets in E'_b to a linear, not necessarily continuous, functional $h: E'_b \rightarrow \mathbb{C}$. Since E is reflexive, it suffices to show that h is continuous.

h is continuous iff $h^{-1}(0)$ is closed in E'_b . The space E is regular, [6; Th. 4], hence $E'_b = \text{projlim } E'_n$ is Frechet. Take a sequence $\{x_n; n = 1, 2, \dots\} \subset h^{-1}(0)$ which converges to x_0 in E'_b . We have to show that $h(x_0) = 0$. Choose $\epsilon > 0$. The set $B = \{x_n; n = 0, 1, 2, \dots\}$ is bounded in E'_b , hence there is $F \in \mathcal{F}$ such that $\sup \{|f(x_n) - h(x_n)|; f \in F, x_n \in B\} < \epsilon$. Fix an $f \in F$ and choose $n \in \mathbb{N}$ so that $|f(x_n) - f(x_0)| < \epsilon$. Then $|h(x_0)| = |h(x_0) - h(x_n)| \leq |h(x_0) - f(x_0)| + |f(x_0) - f(x_n)| + |f(x_n) - h(x_n)| < 3\epsilon$, which implies $h(x_0) = 0$.

CONJECTURE. Regular LB-space may not be sequentially complete.

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