*--INDUCTIVE LIMITS AND PARTITION OF UNITY

V. MURALI

Department of Mathematics
Rhodes University
Grahamstown 6140
South Africa

(Received November 16, 1987 and in revised form September 22, 1988)

ABSTRACT  In this note we define and discuss some properties of partition of unity on *--inductive limits of topological vector spaces. We prove that if a partition of unity exists on a *--inductive limit space of a collection of topological vector spaces, then it is isomorphic and homeomorphic to a subspace of a *--direct sum of topological vector spaces.

KEY WORDS AND PHRASES. Partition of unity, *--inductive limits, *--direct sum.

AMS (Mos) SUBJECT CLASSIFICATION CODES. Primary: 46A12, 46A15.

1. INTRODUCTION

M. De Wilde [1] introduced the concept of partition of unity in an inductive limit space of a family of locally convex spaces which extends the usual partition of unity in function spaces. Around the same time S.O. Iyahen [2] introduced *--inductive limits of topological vector spaces, not necessarily locally convex, as a generalisation of inductive limits. In this paper, we consider the notion of partition of unity in *--inductive limit spaces of topological vector spaces and obtain some useful results some of which are analogous to De Wilde's results in [1]. In section 2, we briefly discuss the well--known concept of F--semi--norms in topological vector spaces. The details may be found in [6]. In section 3, we define the concept of partition of unity in *--inductive limit and using this, obtain a family of F--semi--norms defining the *--inductive limit topology. Finally we conclude with a representation theorem of *--inductive limit space with a partition of unity.

We prove that if a partition of unity exists on a *--inductive limit space of a collection of topological vector spaces, then it is isomorphic and homeomorphic to a subspace of a *--direct sum of topological vector spaces.
2. **F–SEMI–NORMS**

Let $E$ be a vector space over $k$ where $k$ is the field real or complex numbers.

**Definition 2.1**

An F–semi–norm on $E$ is a mapping $\nu : E \to \mathbb{R}$ such that

(i) $\nu(x) \geq 0$ for all $x \in E$;

(ii) $\nu(\lambda x) \leq |\lambda| \nu(x)$ for all $x \in E$ and for all $|\lambda| \leq 1$;

(iii) $\nu(x+y) \leq \nu(x) + \nu(y)$ for all $x, y \in E$;

(iv) for each $x \in E$, $\nu(\lambda x) \to 0$ as $\lambda \to 0$.

Suppose that $V = \{\nu_\alpha : \alpha \in A\}$ is a family of F–semi–norms on $E$. Then $V$ determines a linear topology $\eta$ on $E$. A base of $\eta$–neighbourhoods of the origin in $E$ consists of sets of the form

$$U_{\nu_1, \nu_2, \ldots, \nu_n, \epsilon} = \{x \in E : \nu_{\alpha_j}(x) < \epsilon, j = 1, 2, \ldots, k\}$$

where $\epsilon$ is an arbitrary positive number and $\nu_{\alpha_1}, \nu_{\alpha_2}, \ldots, \nu_{\alpha_n}$ is any finite subcollection of $V$. Also, it is clear that each $\nu_\alpha \in V$ is $\eta$–continuous and $\eta$ is the topology on $E$ determined by the family $Q$ of all $\eta$–continuous F–semi–norms on $E$. In fact, an F–semi–norm $\mu \in Q$ if and only if, for each $\epsilon > 0$ there exists a $\delta > 0$ and a finite collection $\nu_{\alpha_1}, \nu_{\alpha_2}, \ldots, \nu_{\alpha_n}$ of $V$ such that

$$U_{\nu_{\alpha_1}, \nu_{\alpha_2}, \ldots, \nu_{\alpha_n}, \delta} \subseteq \{x : \mu(x) < \epsilon\}.$$

Conversely, we have the following:

**Theorem 2.1**

A vector space topology on $E$ can always be determined by a family of F–semi–norms.

**Proof:** see [6], chapter 1, Proposition 2.

3. **PARTITION OF UNITY:**

Let $(E, \tau)$ be the *–inductive limit of a family of topological vector spaces $(E_i, \tau_i) i \in I$, an index family, relative to linear maps $u_i : E_i \to E$. Suppose further that the index set $I$ is directed and that for each pair indices $i, j \in I$ with $i < j$, there is a continuous linear map $v_{ij} : E_i \to E_j$ such that $u_i = u_j \circ v_{ij}$.

**Definition 3.1** A partition of unity on $E$ is defined to be a family of linear maps $(T_i) (i \in I), T_i : E \to E_i$, which satisfies the following conditions.

(i) $T_i u_j$ is continuous for each pair $(i, j)$.

(ii) For each $j \in I$, $T_i u_j = 0$ except for a finite number of $i \in I$.

(iii) $\sum_{i \in I} u_i T_i$ is the identity map on $E$.

**Remark:** We note that the condition (i) is equivalent to the following condition:

(i) each $T_i : E \to E_i$ is continuous.

**Example 3.2** Suppose $(E, \tau)$ is the inductive limit of locally convex spaces $(E_i, \tau_i)$ $(i \in I)$ with $\{T_i\} (i \in I)$ is a partition of unity of $(E, \tau)$. Then since $\tau$ is coarser than the
\( -\text{inductive limit topology } r^* \) on \( E \), it follows that \( \{ T_i \} (i \in I) \) is also a partition of unity of \((E, r^*)\).

**Example 3.3** Let \( \{ E_n \} \) (\( n = 1, 2, \ldots \)) be a sequence of topological vector spaces, \( E \) be the \(*\)-direct sum of the \(E_n\)'s as defined in [2], and let \( \{ P_n \} \) (\( n = 1, 2, \ldots \)) be the projection maps of \( E \) onto \( E_n \). Then, \( E \) is the \(*\)-inductive limit of the sequence \( \bigoplus_{i=1}^{N} E_i \) \((N = 1, 2, \ldots)\) and the maps \( \{ P_n \} \) constitute a partition of unity.

We now consider some properties of the \(*\)-inductive limit space \((E, v)\) with a partition of unity \( \{ T_i \} (i \in I) \) but first some notations.

For each \( i \in I \), let \( P_i \) be a family of \( F\)-semi-norms on \( E_i \). Then \( P_i \) determines a linear topology \( r_i \) on \( E_i \) and let \( Q_i = \{ v_i^a : a \in I, \} \) be the family of all \( r_i \)-continuous \( F\)-semi-norms on \( E_i \). For each collection of \( F\)-semi-norms \( \{ v_i^a : v_i^a \in Q_i \} \) \((i \in I)\) and each set \( \sigma \) of positive real numbers \( \{ c_i \} \), we define a non-negative real-valued function \( s^\sigma \) on \( E \) by the equation

\[
s^\sigma(x) = \sum_{i \in I} c_i v_i^a(T_i x) \quad \text{for } x \in E. \tag{3.1}
\]

It is easy to verify that \( s^\sigma \) is a well-defined, \( F\)-semi-norm on \( E \). By \( II \) we denote the family of all such \( F\)-semi-norms \( s^\sigma \) for every collection of \( \sigma \) and \( s \).

**Theorem 3.4** The \(*\)-inductive limit topology \( v \) on \( E \) is given by the family \( II \) of \( F\)-semi-norms \( s^\sigma \) defined by the equation 3.1.

**Proof** Let \( \tau_{II} \) be the linear topology on \( E \) generated by the collection \( II \). We have to prove that \( v \geq \tau_{II} \). We will do this in two steps. First, to prove that \( \tau_{II} \) is coarser than \( v \), it is sufficient to show that each \( u_j : (E_j, \tau_j) \to (E, \tau_{II}) \) is continuous. See [4]. Now each \( u_j \) is continuous, and if and only if for any \( s^\sigma \in II \), \( s^\sigma u_j : E_j \to \mathbb{R} \) is continuous.

In fact, for each \( x \in E_j \),

\[
s^\sigma(x) = \sum_{i \in I} c_i v_i^a(T_i x).
\]

But \( T_i x \) is equal to 0 except for a finite number of indices \( i \in I \).

Let \( J = \{ i \in I : T_i x \neq 0 \} \).

Now each \( T_i x \) is continuous from \( E_j \) into \( E_i \), and so \( v_i^a(T_i x) \) is \( \tau_j \)-continuous. Thus we can write \( s^\sigma u_j = \sum_{i \in J} c_i (v_i^a(T_i x)) \) and so \( s^\sigma u_j \) is continuous. From that it follows that \( \tau_{II} \subseteq v \).

For each \( x \in E_i \),

\[
\nu(x) = \nu \left[ \sum_{i \in I} u_{i \circ T_i x} \right] \\
\leq \sum_{i \in I} \nu(u_{i \circ T_i x}).
\]
Now $\nu_{\mathcal{U}_i}$ is a $\tau_i$-continuous $F$-semi-norm on $E_i$ and so belongs to $Q_i$. Hence

$$\nu(x) \leq \sum (\nu_{\mathcal{U}_i})(T_i x) = \tau^S_{\sigma'}(x).$$

Here $s = \{\nu_{\mathcal{U}_i}\} (i \in I)$, and $c_i = 1$ for each $i \in I$. This implies that the identity map $(E, \tau^S_{\sigma'}) \rightarrow (E, \tau)$ is continuous and so $\tau$ is coarser than $\tau^S_{\sigma}$ as required. This completes the proof.

**Corollary 3.5** If each $E_i$ $(i \in I)$ is separated, then $(E, \tau)$ is separated.

**Theorem 3.6** If $B$ is a bounded set in $E$, then $T_i b = 0$ except for a finite number of indices $i \in I$. Hence $B$ is bounded in $E$ if and only if there exists a continuous linear mapping $T$ from $E$ onto some $E_i$ such that $B = u_i T B$.

The proof is analogous to that of the corresponding result in ([1], p3) and so is omitted here.

**Corollary 3.7** If each $\{E_i\}$ is sequentially complete, then $E$ is sequentially complete.

**Proof:** Let $\{x_n\}$ be a Cauchy sequence in $E$. Then $\{x_n\}$ is a bounded set in $E$, and so, by theorem 3.6, there exists a continuous linear mapping $T$ from $E$ into some $E_i$ such that $\{x_n\} = u_i T\{x_n\}$.

Since a continuous linear mapping from one topological vector space into another takes Cauchy sequences to Cauchy sequences, $T\{x_n\}$ is a Cauchy sequence in $E_i$. Now $E_i$ is sequentially complete, and so $T\{x_n\}$ converges to a point $x$ in $E_i$. Therefore $u_i T\{x_n\}$ converges to $u_i x$, since $u_i$ is a continuous linear mapping. Therefore $\{x_n\}$ converges to a point in $E$. Hence the result.

At present it is not known whether the completeness of each $(E_i, \tau_i)$ implies the completeness of $(E, \tau)$. Lastly we prove that the collection of numbers in $\sigma$ of $\|s\|_{\sigma}$ can be chosen in an economical way. An useful application of this is given in [4].

**Proposition 3.8** Let $\sigma' = \{c_i : c_i \geq 1\}$. If $\|s\|_{\sigma'}$, for various collections of $s$ and $\sigma'$, then $\tau^S_{\sigma'} = \tau^S_{\sigma''}$ where $\tau^S_{\sigma''}$ denote the topology generated by $\|s\|_{\sigma'}$ and $\|s\|_{\sigma''}$ respectively.

**Proof** It is obvious that $\|s\|_{\sigma'} \leq \|s\|_{\sigma''}$ and so it is clear that $\tau^S_{\sigma'}$ is coarser than $\tau^S_{\sigma''}$.

Conversely let $U$ be a $\tau^S_{\sigma'}$-neighbourhood of the origin in $E$. Then $V$ contains a set $V$ of the form

$$V = \{x \in E : \tau^S_{\sigma'}(x) < \epsilon; n = 1, 2, \ldots m; \epsilon > 0\} \text{ where } \tau^S_{\sigma'}(x) = \sum c_i^{(n)} a_i^{(n)} (T_i x)$$

Now let for any real number $r$, $[r]$ denote the greatest integer $\leq r$ then

$c_i^{(n)} < [c_i^{(n)}] + 1$;

and if we denote $\sigma' = \{[c_i^{(n)}] + 1\}$, then it is easy to see that $\tau^S_{\sigma'}(x)$ for all $x \in E$. So we
have $U \supseteq V \supseteq V'$, where $V' = \{ x \in E : \tau_{\rho_n}^n(x) < \epsilon ; n = 1,2,\ldots,m, \epsilon > 0 \}$ is a
$\tau\Pi$-neighbourhood of the origin. Thus we have $\tau\Pi$ is coarser than $\tau\Pi'$, and so $\tau\Pi = \tau\Pi'$.

4. DIRECT SUM

In this section we give an analogue of a representation theorem given by D. Keim in [3].
Let $(E, \tau)$ be the $\ast$-inductive limit of topological vector spaces $(E_i, \tau_i)$ ($i \in I$) relative to
linear maps $u_i : E_i \to E$. Suppose, further that, a partition of unity $\{T_i\}$ is defined on
$(E, \tau)$. Then we have the following representation theorem.

**Theorem 4.1**

$(E, \tau)$ is isomorphic and homeomorphic to a subspace of a
$\ast$-direct sum of topological vector spaces.

**Proof:** Define a linear map $\hat{\tau}$ from $(E, \tau)$ into the $\ast$-direct sum of $E_i$'s as follows:

$$\hat{\tau} : E \to \sum_{i \in I} E_i$$

This mapping is well-defined and one-to-one since $\{T_i\}$ satisfies the conditions (ii) and
(iii) of partition of unity respectively. It is easy to check that $\hat{\tau}$ is a linear map and so, is
an isomorphism. Moreover that $\hat{\tau}$ is continuous is shown as follows.

By condition (ii) of partition of unity, $T_{ik}u_j = 0$ except for a finite number of $i \in I$ and for
each fixed $j \in I$. Let $i_1, i_2, \ldots, i_n$ be the finite number of indices such that $T_{ik}u_j = 0$
for $k = 1,2,\ldots,n$. Then $\hat{\tau}u_j = (\sum_{k=1}^{n} I_{ik} T_{ik})u_j$ where $I_{ik}$ is the injection map of

$$E_{i_k} \to \sum_{i \in I} E_i.$$ 

Now for each $i_k$, $k = 1,2,\ldots,m$, $I_{ik} T_{ik}u_j$ is continuous by condition (i) of
partition of unity and continuity of each $I_{ik}$, $k = 1,2,\ldots,n$. Therefore $\hat{\tau}u_j$ is continuous for
each $j \in I$. Consequently $\hat{\tau}$ is continuous [5], as required.

Conversely, let $\hat{\tau}'$ be a linear map defined by $\hat{\tau}' : \sum_{i \in I} E_i \to E$

$$\hat{\tau}'(x_i) = \sum_{i \in I} u_i(x_i).$$

This is well-defined since $x_i = 0$ except for a finite number of $i \in I$. Moreover, $\hat{\tau}'$ is linear
and $\hat{\tau}' | \hat{\tau}'(E) = \hat{\tau}^{-1}$. Also, $\hat{\tau}' \circ I_j = u_j$ is continuous from $E_j \to E$ for each
$j \in I$. Hence $\hat{\tau}'$ is continuous. Thus $\hat{\tau}$ is an isomorphism and a homeomorphism from $E$
onto $\sum_{i \in I} E_i$.

**References**


