HOW MANY NUMBERS SATISFY THE 3X + 1 CONJECTURE?

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ABSTRACT. Let \( \theta(x) \) be the number of numbers not exceeding \( x \) satisfy the 3X + 1 conjecture. We obtain a system of difference inequalities on functions closely related to \( \theta \). Solving this system in the simplest case, we establish \( \theta(x) > cx^3 / 7 \). This improves a result of Crandall [1].

KEY WORDS AND PHRASES. 3X + 1 conjecture, residue class, difference inequality.

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1. INTRODUCTION.

The famous conjecture of Collatz-Kakutani, also known as the Syracuse or the "3X + 1" problem, claims that the sequence

\[
\alpha_{n+1} = T(\alpha_n) = \begin{cases} 
\frac{3\alpha_n + 1}{2}, & \alpha_n \equiv 1 \pmod{2} \\
\frac{\alpha_n}{2}, & \alpha_n \equiv 0 \pmod{2}
\end{cases}
\] (1.1)

converges to the cycle (1,2) for any \( \alpha_0 \in \mathbb{Z}^+ \).

The following well-known heuristic argument serves as an evidence for its validity. Consider \( T \) as though it were a random walk. It is natural to suppose that odd and even numbers appear independently, with probability 1/2 at each jump.
Then $T(n)(a_0)$ should converge since the mathematical expectation

of $\frac{T(a)}{a}$ is about $\left(\frac{3}{2} \cdot \frac{1}{2}\right)^{1/2} < 1$.

Although this conjecture seems to be intractable at present, some supporting
results have been obtained. An interesting review on this problem can be found in
[2]. In particular, Crandall [1] proved that the conjecture is true for many values
of $a_0$. Namely, set $\mathcal{I}(x) = \left\{ u : T^{(k)}(u) = 1 \text{ for some } k > 0 \text{ and } u < x \right\}$.

Thus, $\mathcal{I}(x)$ is just the number of numbers not exceeding $x$ satisfying the conjecture.

Then Crandall's result is $\mathcal{I}(x) > cx$, for appropriate constants $c, r > 0$. However,
his proof gives a very poor value for $r$, about 0.05.

Here we derive a system of difference inequalities on functions closely related
to $\mathcal{I}$ (Lemma 4). Solving this system in the simplest case, we
establish $\mathcal{I}(x) > cx^{3/7}$. Actually our proof gives a little more, namely:

\[
\text{given any } v \equiv 1 \text{ or } 2 \pmod{3} \text{ that is not in a cycle, for all } x > 1
\]

\[
\left\{ n < v x : T^{(k)}(n) = v \text{ for some } k > 1 \right\} \geq c_0 x^{3/7}
\]

where $c_0$ is a positive constant independent of $v$.

In some sense the proof may be regarded as an attempt to formalize the above
mentioned heuristic argument.

2. RESULTS.

Consider the infinite directed graph $G$ on the vertex set $V = \mathbb{Z}^+$ and the edge
set $E = \{(T(v), v)\}$, whose edges are oriented from $T(v)$ to $v$. Denote by $G(v,x)$ an
induced subgraph of $G$ whose vertex set consists of all integers $n$ such that
some $T^k(n) = v$ and $T^i(n) < x$ for $0 < i < k$. That is, it consists of all integers $n$
whose trajectory hits $v$ and remains below $x$ the entire time. In particular $G(v,x)$ is
the empty set if $x < v$. We also put $G(v) = G(v,\infty)$. Observe that $G(v)$ has at most
one cycle since the in degree of each vertex, but may be $v$, is one. Moreover, if $v$
do not lie in a cycle of $G$ then $G(v)$ is a tree.

Here we prefer to deal with $U$, the mapping inverse to $T$, namely:

\[
U(a) = \begin{cases} 
2a, & a \equiv 0, 1 \pmod{3} \\
2a - \frac{1}{3}, & a \equiv 2 \pmod{3}
\end{cases} \quad (2.1)
\]

Since only numbers $a \equiv 2 \pmod{3}$ have two inverses under $T$, we wish to analyze
iterates under $U = T^{-1}$ restricted to integers $\equiv 2 \pmod{3}$. To do this we must
consider values of $a \pmod{9}$.
Let $S_n$ be a complete system of residue classes modulo $3^n$. We split $S_n$ as follows:

$$S_n = \bigcup_{i=0}^{2} R_i^n, \text{ where } \alpha \in R_i^n \iff \alpha \equiv i \pmod{3}.$$ 

Furthermore, put

$$R_2^2 = Q_5^n \cup Q_8^n, \text{ where } \alpha \in Q_i^n \iff \alpha \equiv i \pmod{9}.$$ 

Obviously, $U: R_n^0 \rightarrow R_n^0$ and $U: R_n^1 \rightarrow R_n^2$. The action of $U$ on $R_n^2$ can be split into the four following operators:

$$U_1: R_n^2 + R_n^2, U_1(\alpha) = 4\alpha$$

$$U_2: Q_5^n + R_n^{0,1}, U_2(\alpha) = \frac{2\alpha - 1}{3}$$

$$U_3: Q_8^n + R_n^{2,1}, U_3(\alpha) = \frac{4\alpha - 2}{3}$$

$$U_4: Q_8^n + R_n^{8,1}, U_4(\alpha) = \frac{2\alpha - 1}{3}$$

The following lemma is an easy exercise in elementary number theory:

**LEMMA 1.**

(i) $U_1$ is a bijection $R_n^2 + R_n^2$. Moreover, if $\alpha \in R_n^2$ then $\ell = 3^{n-1}$ is the smallest positive integer such that $U_1(\ell)(\alpha) = \alpha$.

(ii) $U_3$ is a bijection $Q_8^n + R_n^{2,1}$.

(iii) $U_4$ is a bijection $Q_8^n + R_n^{8,1}$.

The action of $U$ on $R_n^0$ and $R_n^1$ is much simpler. Namely, $U: R_n^0 \rightarrow R_n^0$ and $U: R_n^1 \rightarrow R_n^2$ are bijections. Moreover, since $\alpha \in R_n^0$ implies $U(\alpha) = 2\alpha \in R_n^0$ we get

**LEMMA 2.** If $\nu \in R_n^0$ then $G(\nu)$ is a chain.

Now we define the functions we deal with in this paper.

Let $v \equiv m \pmod{3^n}$. We set $f(v,x) = f_n^m(v,x) = \lvert G(v,x) \rvert$. (The reason for using the redundant notation $f_n^m(v,x)$ instead of $f(v,x)$ is to simplify the statement of the difference inequalities that follow.)

Observe that for $v \leq x$

$$f_n^m(v,x) = 1 + \lfloor \log_2 \frac{x}{v} \rfloor, \text{ m } \in R_n^0, \quad (2.2)$$

$$f_n^m(v,x) = 1 + f_n^{2m}(2v,x), \text{ m } \in R_n^1. \quad (2.3)$$
Furthermore, let \( W = \{ w \} \) be the set of those vertices of \( G \) which do not belong to a cycle. For instance, \( u^k(4) \in W \) for all \( K > 0 \). Then \( G(w) \) is a tree and we set

\[
\phi_n^m(y) = \inf \{ \phi^m_n(v, 2^y v) : v \in W, v \equiv m \pmod{3^n} \}.
\]

Note that for any \( m \equiv 2 \pmod{3} \) and \( n \), the set \( \{ v : v \in G(u), v \equiv m \pmod{3^n} \} \neq \emptyset \) because \( 2^k v \) is in this set and 2 is a primitive root \( \pmod{3^n} \) for all \( n \).

**Lemma 3.** \( \phi_n^m(y) \) is nondecreasing function of \( y \).

**Proof.** Obviously, \( \phi_n^m(v, x) \) is a nondecreasing function of \( x \).

Hence, \( \phi_n^m(y) = \inf \phi_n^m(v, 2^y v) \) is nondecreasing function of \( y \).

The following lemma gives important recurrent inequalities on \( \phi_n^m(y) \).

**Lemma 4.** For \( y > 0 \),

\[
\phi_n^m(y) > \phi_n^m(y - 2) + \phi_n^{m+2}(y + \alpha - 2), \quad m \in \mathbb{Q}^n_2
\]

\[
\phi_n^m(y) > \phi_n^m(y - 2) + \phi_n^{m+2}(y + \alpha - 1), \quad m \in \mathbb{Q}^n_8
\]

where \( \alpha = \log_2 3 = 1.585 \) and

\[
\phi_n^{m+3n-1}(y) = \min (\phi_n^m(y), \phi_n^{m+2n-1}(y), \phi_n^{m+2n-1}(y)).
\]

**Proof.** (2.5) follows immediately from the definition of \( \phi_n^m(y) \). Let us demonstrate (2.4). If \( v \equiv m \pmod{3^n} \), \( m \in \mathbb{Q}_n^8 \) then, by (2.1), if \( v < x \),

\[
|G(v, x)| > |G(4v, x)| + G\left(\frac{2v - 1}{3}, x\right).
\]

If \( \frac{2v - 1}{3} \equiv 0 \pmod{3} \) then \( G\left(\frac{2v - 1}{3}, x\right) \) is a chain by lemma 2. Thus, by (2.2), if \( v < x \),

\[
f_n^m(v, x) = f_n^m(4v, x) + 1 + \left[ \log_2 \frac{3x}{2v - 1} \right].
\]

Hence, \( \phi_n^m(y) > \phi_n^m(y - 2) + [y + \alpha] \).

If \( m \in \mathbb{Q}_n^8 \) then \( G(v, x) \) is a forest. Hence,

\[
|G(v, x)| = |G(4v, x)| + G\left(\frac{2v - 1}{3}, x\right).
\]
By \( \frac{2v - 1}{3} < \frac{2v}{3} \) and by Lemma 3 we get, if \( y > 0 \) and \( x = 2^v y \), then

\[
\psi_n^v(y) = \inf \psi_n^m(v, x) = \inf \left( \frac{2^v - 1}{3} \psi_{n-1}^m \left( \frac{2^v - 1}{3}, x \right) \right) >
\]

\[
> \inf \psi_n^m(4^v, x) + \inf \left( \frac{2^v - 1}{3} \psi_{n-1}^m \left( \frac{2^v - 1}{3}, x \right) \right) > \phi_n^m(y - 2) + \phi_n^m(y + a - 1).
\]

The case \( m \in \mathbb{Q}_n \) may be considered similarly to the case \( m \in \mathbb{Q}_n^8 \). We omit the details.

**Theorem 1.** \( \theta(x) > c_2x^{\frac{3}{7}} \).

**Proof.** For \( n = 2 \) the system (2.4) becomes for \( y > 0 \),

\[
\phi_2^2(y) > \phi_2^8(y - 2) + \phi_1^5(y + a - 2),
\]

\[
\phi_2^8(y) > \phi_2^5(y - 2) + \phi_1^5(y + a - 1),
\]

\[
\phi_2^5(y) > \phi_2^5(y - 2),
\]

where \( \phi_1^5(y) = \min (\phi_2^2(y), \phi_2^8(y), \phi_2^5(y)) \). Observe that \( \phi_2^8(y) > \phi_1^5(y) \) for \( y > 2 \) by

\[
\phi_2^8(y) > \phi_2^5(y - 2) + \phi_1^5(y + a - 1) > \phi_1^5(y),
\]

since \( \phi_1^5(y + a - 1) > \phi_1^5(y) \) and \( \phi_2^5(y - 2) > 0 \) if \( y > 2 \). Hence,

\[
\phi_1^5(y) = \min (\phi_2^2(y), \phi_2^8(y)) > \min (\phi_2^2(y), \phi_2^2(y - 2)) = \phi_2^2(y - 2).
\]

This yields if \( y > 6 \),

\[
\phi_2^2(y) > \phi_2^5(y - 4) + \phi_1^5(y + a - 1) + \phi_1^5(y + a - 2)
\]

\[
> \phi_2^5(y - 6) + \phi_1^5(y + a - 1) + \phi_1^5(y + a - 2)
\]

\[
> \phi_2^5(y - 6) + \phi_2^5(y + a - 5) + \phi_2^5(y + a - 4).
\]

The initial conditions \( \phi_2^2(0) = 1 \) imply \( \psi_2^2(y) > 1 \) for \( y > 6 \), whence one proves by induction on \( n \), that for \( n < y < n + 1 \), one has \( \psi_2^2(y) > c_1 \lambda^y \), where \( \lambda = 1.3534 \) is the largest root of \( 1 - \lambda^{-6} + \lambda^{-5} + \lambda^{-4} \).

Finally, we obtain \( \theta(x) > c_2x^{\frac{\log_2 \lambda}{\frac{3}{7}}} = c_2x^{\frac{3}{7}} \), where \( \log_2 \lambda = 0.436 \).

**Remark.** Although system (2.4) seems to be very complicated and we were unable to
solve it for \( n > 3 \), averaging it over all residue classes modulo \( 3^n-1 \) looks much more attractive. Namely, define

\[
F_n(y) = 3^{-n+1} \sum_{m \in \mathbb{Z}/3^n} \phi^m_n(y).
\]

Using lemmas 1 and 4 we get

\[
3^{n-1} F_n(y) = \sum_{m \in \mathbb{Z}/n} \phi^m_n(y-2) + \sum_{m \in \mathbb{Z}/n} \phi^m_n(y+\alpha-2) + \sum_{m \in \mathbb{Z}/n} \phi^m_{n-1}(y+\alpha-1) = 3^{n-1} F_n(y-2) + 3^{n-2} F_{n-1}(y + \alpha - 2) + 3^{n-2} F_{n-1}(y + \alpha - 1).
\]

Thus,

\[
F_n(y) > F_n(y-2) + \frac{1}{3} F_{n-1}(y + \alpha - 2) + \frac{1}{3} F_{n-1}(y + \alpha - 1).
\]

Observe that the associated limit equation \( 1 = \lambda^{-2} + \frac{1}{3} (\lambda^{\alpha-2} + \lambda^{\alpha-1}) \) has \( \lambda = 2 \) as the smallest positive root. Therefore, one might expect that the solution of the difference inequalities gives \( \theta(x) > c_n x^n \), where \( r_n + 1 \) when \( n \) tends to infinity.

REFERENCES

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