ABSTRACT. The Lyapunov mapping on $n \times n$ matrices over $\mathbb{C}$ is defined by $L_A(X) = AX + XA^*$; a matrix is stable if all its characteristic values have negative real parts; and the inertia of a matrix $X$ is the ordered triple $\text{In}(X) = (\pi, \nu, \delta)$ where $\pi$ is the number of eigenvalues of $X$ whose real parts are positive, $\nu$ the number whose real parts are negative, and $\delta$ the number whose real parts are 0. It is proven that for any normal, stable matrix $A$ and any hermitian matrix $H$, if $\text{In}(H) = (\pi, \nu, \delta)$ then $\text{In}(L_A(H)) = (\nu, \pi, \delta)$. Further, if stable matrix $A$ has only simple elementary divisors, then the image under $L_A$ of a positive-definite hermitian matrix is negative-definite hermitian, and the image of a negative-definite hermitian matrix is positive-definite hermitian.

KEY WORDS AND PHRASES. Lyapunov, stable matrix, matrix inertia, positive-definite matrix

1980 AMS SUBJECT CLASSIFICATION CODES. 15A18, 15A42

For many years stable matrices have interested applied mathematicians because, for a system of linear homogeneous differential equations whose coefficients are constant, a stable matrix of coefficients is a necessary and sufficient condition that the solution be asymptotically stable. Recently, algebraists too have become interested in stable matrices.

Definition: A square matrix is stable if all its characteristic values have negative real parts.

(In this article, the entries of all matrices are complex numbers unless stated otherwise.)

A classical test for stability of matrices is Lyapunov's theorem, whose statement is facilitated by some notation:

- $S$ = set of all $n \times n$ stable matrices
- $H$ = set of all $n \times n$ hermitian matrices
- $iH$ = set of all $n \times n$ skew-hermitian matrices
- $G$ = set of all $n \times n$ positive-definite hermitian matrices
- $N$ = set of all $n \times n$ negative-definite hermitian matrices
- $L_A(X) = AX + XA^*$, where $A$ and $X$ are $n \times n$ matrices and $A^*$ is the conjugate transpose of $A$.

(It is trivial to verify that $L_A(\cdot)$, the Lyapunov mapping, is a linear transformation on the linear space $M_n$ of $n \times n$ matrices.)

Lyapunov's theorem is usually expressed as statement a) of

Theorem 1: The following three statements are equivalent:

a) $A \in S$ = there exists $G \in \Pi$ such that $L_A(G) = -I$;
b) A ∈ S implies for every G ∈ N, there exists G ∈ Π such that \( I_A(G) = G \) \( G \) there exists \( G \) \( N \) and there exists \( G \) \( \Pi \) such that \( I_A(G) = G \). [Taussky, 1964; p. 6, thms 2-3];

c) Let \( C = A + S \) (a real and < 0, S \( I H \)) and \( D = \text{diag}(d_1, \ldots, d_n) \) with \( d_i \) real \( (i=1, \ldots, n) \). Then \( CD \in S \Rightarrow d_i > 0 \) for all \( i \). [Taussky, 1961, J. Math Anal. & App.]

The equivalences are proven (essentially) in Taussky's articles. An analytic proof a) is in Bellman, pp. 242-245, and a topological proof in Ostrowski & Schneider.

Theorem 1 suggests that the operator \( I_A(\cdot) \) might give rise to other tests for stability; such usefulness is limited, however, by the following

**Theorem 2:** The range of \( I_A(H) \) as a function of \( H \in \Pi \) and \( A \in S \) is that subset of \( H \) with \( \nu \neq 0 \) (where \( \nu \) denotes the number of characteristic vectors with negative real parts). [Stein, p. 352, thm 2].

Some useful theorems result if further restrictions are imposed on \( A \) besides stability. These theorems are obtained via a topological route and require additional concepts.

**Definition:** The **inertia** of an \( n \times n \) matrix \( X \) is the ordered triple of integers \((\pi(X), \nu(X), \delta(X)) = \text{In}(X)\) where \( \pi(X) \) is the number of characteristic values of \( X \) whose real parts are positive, \( \nu(X) \) the number whose real parts are negative, and \( \delta(X) \) the number whose real parts are 0. If \( n \times n \) matrices \( M \) and \( N \) possess the same inertia, this will be denoted by \( M \equiv N \).

Let \( M \) and \( N \) be \( n \times n \) hermitian matrices. \( M \) and \( N \) are congruent (denoted \( M \equiv N \)) \( \exists P \) non-singular such that \( M = P^*NP \).

Recall that all norms in the set of all \( n \times n \) matrices \( M_n \) induce the same topology. In \( M_n \) so topologized, matrices \( M \) and \( N \) are connected \( \exists \) there exists a connected set containing both \( M \) and \( N \). The relationship of being connected is an equivalence relation, which will be denoted by \( \equiv \). \( M \) and \( N \) are arc-wise connected \( \exists \) there exists a continuous function \( f \) from the real interval \([0,1]\) into \( M_n \) such that \( f(0) = M \) and \( f(1) = N \). This, too, is an equivalence relation in \( M_n \) and will be denoted by \( \sim \).

The preceding concepts are brought together by the following theorem:

**Theorem 3:** In the set \( N_n \) of all non-singular \( n \times n \) matrices with the relative topology induced by any norm, \( A \equiv B \) and \( A \equiv B \) \( (\forall A, B \in N_n) \).

[Schneider; pp. 818-819, lemmata 1 & 2]. Let \( H \equiv P \) denote the set of all \( n \times n \) hermitian matrices of rank \( r \). In \( H \equiv P \) with the relative topology induced by any norm the four equivalence relations \( \equiv, \equiv, \equiv, \equiv \) coincide. [Schneider; p. 820].

The relationship between algebraic features of hermitian matrices and topological features expressed by theorem 3 makes it possible to discover the variation in signature induced by the Lyapunov mapping \( I_A(\cdot) \) whenever \( A \in S \) is normal and \( H \in H \).

**Theorem 4:** If \( A \in S \) is normal, then for any \( H \in H \) with \( \text{In}(H) = (\pi, \nu, \delta) \), \( \text{In}(I_A(H)) = (\nu, \pi, \delta) \).

**Proof:** Let \( A \in S \) be normal, \( \{a_i\} \equiv H \) be its characteristic values, \( H \in H \), \( \text{In}(H) = (\pi, \nu, \delta) \), and \( I_A(H) = AH + HA^* = C \).

Since \( A \) is normal, it is unitarily similar to a diagonal matrix: \( VAV^* = \text{diag}(a_1, \ldots, a_n) \), \( V \) unitary. Also a basis for \( n \)-dimensional space can be
formed from the characteristic vectors of $A$, $(a_i)^T$.

For any $i$, $a_i C - a_i(A^H H A^*) = a_i a_i H + a_i H A^* - a_i H(a_i I + A^*) = \text{rank of } H$ (since $a_i I + A^*$ is non-singular, for the characteristic values of $-A^*$ are $(-a_i)^T$ and $(a_i)^T \cap (-a_i)^T = \emptyset$ because real part of $a_i$ = real part of $a_i < 0$ (i=1,...,n)). Therefore, rank $(H) = \text{rank } (Z_A(H))$.

Because $Z_A$ is a linear transformation of $M_n$ onto itself, it is continuous. If $Z_A$ is restricted to $H \subseteq M_n$ it is continuous and onto $H$. Therefore, $Z_A$ maps topologically connected components of $H$ onto components of $H$ since rank is preserved by $Z_A$. But by theorem 3 topologically connected components coincide with inertial components. Therefore, $Z_A$ maps $\text{In}(H)$ on $\text{In}(C)$.

$H \in H$ and since $VHV^*$ is congruent to $H$, $\text{In}(VHV^*) = \text{In}(H)$. Hence, $\text{In}(Z_A(VHV^*)) = \text{In}(Z_A(H)) = \text{In}(C)$.

Let $D = Z_A(VHV^*) - A(VHV^*) + (VHV^*)A^*$. Then $V^*DV = (V^*AV)H + H(V^*A^*V).$ Because $D \in H$, $\text{In}(Z_A(VHV^*)) = \text{In}(Z_A(VHV^*)) = \text{In}(C)$. $Z_A(VHV^*)(K)$ is of the form $\text{diag}(a_1, a_2, ..., a_n) = \text{diag}(a_1, a_2, ..., a_n)$, where $R(a)$ denotes the real part of complex number $a$. Let $\mu$ the mxm identity matrix, and $O_n$ the m$x$m$O_n$ matrix. Since $R(a_i) < 0$ (i=1,...,n), $\text{In}(Z_A(VHV^*)(K)) = (\mu, \pi, \delta)$. Therefore, $\text{In}(C) = (\mu, \pi, \delta)$.

QED

The preceding theorem was based on the unitary similarity of $A$ to a diagonal matrix; this property was used first to show the invariance of rank and then to display the inertia when both $A$ and $H$ were expressed in canonical form. The next theorem generalizes the last in that $A$ need be similar (not unitarily similar) to a diagonal matrix, but it is more restrictive of the inertia of $H$.

**Theorem 5:** If $A \in S$ has only simple elementary divisors, then $Z_A(H) = N$ and $Z_A(N) = I$.

**Proof:** Since $A$ has only simple elementary divisors, it is similar to a diagonal matrix. As in the proof of the preceding theorem, rank $(H) = \text{rank } (Z_A(H))$. Likewise, $Z_A$ maps $\text{In}(H)$ on $\text{In}(Z_A(H))$. By Lyapunov's theorem (1a), $\exists H \in Z_A(H) : Z_A(H) = -I \in N$. Therefore, $Z_A(N) \subseteq N$. But by the alternative version (1b) of Lyapunov's theorem, $N \subseteq Z_A(N)$.

Therefore, $Z_A(N) = I$.

QED

**REFERENCES**


4. Stein, P. "On the Ranges of Two Functions of Positive Definite

