A CHARACTERIZATION OF THE ALGEBRA OF HOLOMORPHIC FUNCTIONS
ON A SIMPLY CONNECTED DOMAIN

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ABSTRACT: Let A be a singly-generated $\mathcal{F}$-algebra. It is shown that A is isomorphic to $H(\Omega)$ where $\Omega$ is a simply connected domain in $\mathbb{C}$ if and only if A has no topological divisors of zero. It follows from this that there are exactly three $\mathcal{F}$-algebras (up to isomorphism) which are singly generated and have no topological divisors of zero.

KEY WORDS AND PHRASES. $\mathcal{F}$-algebras, holomorphic functions, topological divisors of zero
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1. INTRODUCTION.

The algebra $H(\Omega)$ of holomorphic functions on a domain $\Omega \subseteq \mathbb{C}$ with pointwise operations and compact-open topology is an interesting example of an $\mathcal{F}$-algebra. This algebra has been characterized in terms of some of the special properties it enjoys that are derived from the fact that it consists of holomorphic functions. (See for example [1], [2], [3], [4] and [5] for characterizations in terms of the local maximum modulus principle, the Cauchy estimate, Montel's theorem, the existence of derivations, and Taylor's theorem.) In [6] a characterization of the algebra of entire functions in terms of Liouville's theorem is given.

Watson [5] shows that an $\mathcal{F}$-algebra A which has a Schauder basis that is generated by an element $z \in A$ with open spectrum is algebraically and topologically isomorphic to $H(\Omega)$ where $\Omega$ is an open disk in $\mathbb{C}$. In this paper we study $\mathcal{F}$-algebras that are generated by a single element $z$ (without requiring that $z$ generate a basis for A). Of course, this condition alone is not enough to completely describe the algebra $H(\Omega)$ among $\mathcal{F}$-algebras. We will show, however, that this together with the condition that A has no topological divisors of zero, completely characterizes $H(\Omega)$ for a simply connected domain $\Omega$. It follows from this that there are exactly three singly generated $\mathcal{F}$-algebras (up to isomorphism) which have no topological divisors of zero.

2. PRELIMINARIES.

An $\mathcal{F}$-algebra is a complete metrizable locally m-convex algebra. (All the algebras we consider are assumed to be commutative algebras over $\mathbb{C}$.) The topology of such an algebra is given by an increasing sequence of seminorms $\{p_n : n \in \mathbb{N}\}$. Each $p_n$ determines a Banach algebra $A_n$ which is the completion of $A/\ker(p_n)$. If $n \leq m$ then the natural homomorphism
from $A/\ker(p_m)$ to $A/\ker(p_n)$ induces a norm decreasing homomorphism $\tau_{nm}: A_m \rightarrow A_n$ whose range is a dense subalgebra of $A_n$. The Banach algebras $A_n$ with maps $\tau_{nm}$ form an inverse limit system and $\lim_n (A_n, \tau_{nm})$ is topologically and algebraically isomorphic to $A$.

The maximal ideal space of $A$ is the space $\mathfrak{m}(A)$ consisting of all non-zero continuous multiplicative linear functionals on $A$ endowed with the Gelfand topology. This topology is the weak topology on $\mathfrak{m}(A)$ generated by the Gelfand transforms $\hat{x}: \mathfrak{m}(A) \rightarrow \mathbb{C}$ defined by $\hat{x}(f) = f(x)$. The map $\gamma: A \rightarrow \hat{A}$ is a continuous homomorphism onto the algebra $\hat{A} \subset C(\mathfrak{m}(A))$ of Gelfand transforms. For each $n \in \mathbb{N}$ the quotient map $\tau_n$ from $A$ onto $A/\ker(p_n)$ induces a homeomorphism $\tau_n^*\mathfrak{m}(A)$ of the maximal ideal space $\mathfrak{m}(A_n)$ of $A_n$ onto a compact subset $M_n$ of $\mathfrak{m}(A)$. For $n \leq m$ we have $M_n \subset M_m$ and $\mathfrak{m}(A) = \bigcup M_n$.

The spectrum of $z \in A$ is the set $\sigma = \sigma(z) = \{f(z) \mid f \in \mathfrak{m}(A)\}$. For each $n \in \mathbb{N}$ the set $\sigma_n = \sigma_n(z) = \{f(z) \mid f \in M_n\}$ and $\sigma = \bigcup \sigma_n$. The element $z \in A$ generates $A$ if $A$ is the smallest closed subalgebra containing $z$ and $e$ (the identity of $A$). In this case the spectrum map $\varphi: \mathfrak{m}(A) \rightarrow \sigma(z)$ defined by $f \mapsto f(z)$ is a continuous bijection $[7]$.

An element $z$ in a Banach algebra $B$ is a topological divisor of zero if the multiplication map $M_z: A \rightarrow zA$ is not an isomorphism (i.e., does not have a continuous inverse). In an $\mathcal{F}$-algebra $A$, $z$ is called a topological divisor of zero if for each sequence $(p_n: n \in \mathbb{N})$ of seminorms defining the topology of $A$ there exists $k \in \mathbb{N}$ such that $\tau_k(z)$ is a topological divisor of zero in the Banach algebra $A_k$ $[8$, pp. 46-47$].

3. CHARACTERIZING $H(\Omega)$

Let $\Omega \subset \mathbb{C}$ be a simply connected domain. The algebra $H(\Omega)$ of holomorphic functions on $\Omega$ is an $\mathcal{F}$-algebra in the compact-open topology. It is well known that $H(\Omega)$ has no (nonzero) topological divisors of zero $[9]$, and is singly-generated. We will show that these last two properties of $H(\Omega)$ completely characterize it among $\mathcal{F}$-algebras.

For the rest of this paper $A$ will denote an $\mathcal{F}$-algebra with identity $e$ which is generated by $z$, where $z$ is not a scalar multiple of $e$, and which has no nonzero topological divisors of zero.

**Lemma 1.** $A$ is semisimple and so the Gelfand transform is a bijection.

**Proof:** Suppose $y \in \text{Rad}(A)$, $y \neq 0$. Then $\sigma(y) = \{0\}$ and by $[8$, Proposition 11.8$]$ $y$ is a topological divisor of zero.

**Lemma 2.** The spectrum $\sigma(z)$ is a domain in $\mathbb{C}$.

**Proof:** If $\lambda \in \sigma(z)$ is a boundary point of $\sigma(z)$, then again by $[8$, Proposition 11.8$]$, $z - \lambda e$ is a topological divisor of zero. Thus $\sigma(z)$ is open.

If $\sigma(z)$ includes the two components $U_1$ and $U_2$, then the characteristic functions $h_1$ of $U_1$ and $h_2$ of $U_2$ are analytic on $\sigma(z)$. By the functional calculus there exist $x_1, x_2 \in A$ with $\tilde{x}_1 = h_1(\tilde{z})$ and $\tilde{x}_2 = h_2(\tilde{z})$. Clearly $\tilde{x}_1 \tilde{x}_2 = 0$ so by Lemma 1 $x_1 x_2 = 0$ and thus these elements are nonzero (topological) divisors of zero.

**Lemma 3.** The domain $\sigma(z)$ is simply connected.

**Proof:** Let $\varphi: \mathfrak{m}(A) \rightarrow \sigma(z)$ be the spectrum map and for $t \in \sigma(z)$ we use the notation $f_t = \varphi^{-1}(t)$. For each $x \in A$ define $\tilde{x}: \sigma(z) \rightarrow \mathbb{C}$ by $\tilde{x}(t) = \tilde{x}(\varphi^{-1}(t)) = \varphi(f_t)$. We show that $\tilde{x}$ is analytic on $\sigma(z)$. Since $A$ is generated by $z$ there exists a sequence of polynomials $p_n = p_n(z)$ converging to $x$ in $A$ and so $f(p_n(z)) \rightarrow f(x)$, for every $f \in \mathfrak{m}(A)$. For $t \in \sigma(z)$, $p_n(t) = p_n([f_t(z)]) = f_t(p_n(z))$ converges to $f_t(x) = \tilde{x}(f_t)$, so each $\tilde{x}$ is a pointwise limit of polynomials on $\sigma(z)$. We now show that this convergence is uniform on
compact subsets of $\sigma(z)$ and so each $\tilde{x}$ is analytic on this spectrum. Since $A$ has no topological divisors of zero, for each $n \in \mathbb{N}$ there exists $m_n$ such that $\sigma_n \subseteq \text{int} \sigma_m$ (see Arens [9]). So without loss of generality, we may assume that for $n = 1, 2, \ldots$,

$$\sigma_n \subseteq \text{int} \sigma_{n+1} \subseteq \sigma_{n+1}$$

and $\sigma = \bigcup \text{int} \sigma_n = \bigcup \sigma_n$.

Since $\varphi|\mathcal{M}_n$ is a homeomorphism onto its image $\varphi(\mathcal{M}_n) = \sigma_n$, it follows that if $K$ is a compact subset of $\sigma$ there exists $n \in \mathbb{N}$ such that $K \subseteq \text{int} \sigma_n \subseteq \sigma_n$. Thus $\varphi^{-1}(K) \subseteq \varphi^{-1}(\sigma_n) = \mathcal{M}_n$ and so $\varphi^{-1}(K)$ is a compact subset of $\mathcal{N}(A)$. Now $p_n \to x$ by the continuity of the Gelfand map $\gamma$, $p_n \to \tilde{x}$ in $\tilde{A}$, i.e., the convergence is uniform on compact subsets of $\mathcal{N}(A)$. Thus for $\varepsilon > 0$ and sufficiently large $n$,

$$|p_n(f) - \tilde{x}(f)| < \varepsilon$$

for $f \in \varphi^{-1}(K)$, which is the same as

$$|p_n(t) - \tilde{x}(t)| < \varepsilon$$

for $t \in K$. Thus each $\tilde{x}$ is the limit of polynomials, uniformly on compact subsets of $\sigma(z)$, and hence is analytic there.

Let $h \in H(\sigma(z))$. We show that $h$ is the limit of polynomials in $H(\sigma(z))$, then it follows that $\sigma(z)$ is simply connected. Using the functional calculus for $\mathcal{S}$-algebras we find $x \in A$ such that $\tilde{x}(f) = h(\tilde{x}(f))$, $f \in \mathcal{N}(A)$. Therefore $h = \tilde{x}$. This together with the preceding paragraph completes the proof.

**Lemma 4.** $\sigma(z)$ is homeomorphic with $\mathcal{N}(\lambda)$.

**Proof:** The map $\varphi$ is a continuous bijection. But $x \circ \varphi^{-1} = \tilde{x}$ is continuous for each $\tilde{x} \in \tilde{A}$ and so the continuity of $\varphi^{-1}$ follows from the fact that the topology of $\mathcal{N}(A)$ is the weak topology generated by $\lambda$.

Lemma 4 may also be derived from [7, Theorem 1.3]. Notice that Lemmas 3 and 4 imply that $\mathcal{N}(A)$ is homeomorphic to the open unit disc. We now prove our main result.

**Theorem 1.** An $\mathcal{S}$-algebra $A$ is algebraically and topologically isomorphic to $H(\Omega)$ for a simply connected domain $\Omega$ if and only if $A$ is singly-generated and has no nonzero topological divisors of zero.

**Proof:** That the $\mathcal{S}$-algebra $H(\Omega)$ has these properties is discussed at the beginning of this section.

Conversely, let $\tilde{A} = (\tilde{x} | x \in A)$ and equip $\tilde{A}$ with the compact open topology. From the proof of Lemma 3, $\tilde{A} = H(\sigma(z))$ algebraically and topologically. Also, $\tilde{A}$ and $\tilde{A}$ are isomorphic as $\mathcal{S}$-algebras via the map $\delta: \tilde{A} \to \tilde{A}$ by $\tilde{x} \to \tilde{x} \circ \varphi^{-1}$. Since the Gelfand map $\gamma: A \to \tilde{A}$ is bijective by Lemma 1, it follows that the map $\delta \circ \gamma$ is a continuous bijection of $A$ onto $\tilde{A} = H(\sigma(z))$. The open mapping theorem now yields the result.

The notion of topological divisor of zero we used above is that due to Michael [8, p. 47]. Our Theorem 1 does not remain valid if that notion is replaced by the stronger definition of Arens [10] (called strong topological divisor of zero by Michael). In fact, the $\mathcal{S}$-algebra $\mathcal{C}[X]$ of formal power series (with the topology of pointwise convergence in the coefficients) is singly generated and has no strong topological divisors of zero [11]. But this algebra is not isomorphic to $H(\Omega)$ for any domain $\Omega$.

The Riemann mapping theorem yields the following corollary:

**Corollary 1.** There are (up to isomorphism) exactly three $\mathcal{S}$-algebras which are singly generated and have no nonzero topological divisors of zero. Namely, $\mathbb{C}$, the algebra $H(D)$ where $D$ is the open unit disk, and the algebra $\mathcal{S}$ of entire functions.
Brltel [6] (see also [12] and [13]) gave a characterization of the algebra of entire functions as a singly-generated Liouville algebra without topological divisors of zero. A Liouville algebra is an $\mathcal{F}$-algebra in which every element with bounded spectrum is a scalar multiple of the identity. We give another proof of Brltel's theorem based on our Theorem 1.

**THEOREM 2.** (Brltel) An $\mathcal{F}$-algebra $A$ is topologically and algebraically isomorphic to the algebra $\mathcal{E}$ of entire functions if and only if $A$ is a singly-generated Liouville algebra with no nonzero topological divisors of zero.

**PROOF:** By Theorem 1, $\sigma(z)$ is simply connected and $A$ is isomorphic to $H(\sigma(z))$. If $\sigma(z) \neq \mathbb{C}$ then there is a one-to-one analytic function $\psi$ from $\sigma(z)$ onto $\mathbb{D}$. Thus there exists $x \in A$ such that $\hat{x} = \psi \cdot \hat{z}$. Clearly $x$ is not a scalar multiple of $\hat{z}$ and $\sigma(x) = D$, contradicting the assumption that $A$ is Liouville.

A natural extension of the notion of a simply connected domain to $\mathbb{C}^n$ is that of a Runge domain. If $\Omega$ is a Runge domain in $\mathbb{C}^n$ then $H(\Omega)$ is n-generated and has no nonzero topological divisors of zero. We pose the question of whether a finitely-generated $\mathcal{F}$-algebra $A$ with no nonzero topological divisors of zero is isomorphic to $H(\Omega)$ for a Runge domain $\Omega$. In the case that $A$ has a finitely-generated Schauder basis in which the joint spectrum of the generators is an open set in $\mathbb{C}^n$, it is shown in [14] that $A$ is isomorphic to $H(\Omega)$ for a complete logarithmically convex Reinhardt domain $\Omega$.

**REFERENCES**

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