A NOTE ON RINGS WITH CERTAIN VARIABLE IDENTITIES

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ABSTRACT. It is proved that certain rings satisfying generalized-commutator constraints of the form \([x^m, y^n, y^n, \ldots, y^n] = 0\) with \(m\) and \(n\) depending on \(x\) and \(y\), must have nil commutator ideal.

KEY WORDS AND PHRASES. Commutator ideal, periodic ring.

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1. INTRODUCTION.

Let \([x_1, x_2]\) denote \(x_1 x_2 - x_2 x_1\), and for \(k > 2\), let \([x_1, x_2, \ldots, x_k] = [x_1, \ldots, x_{k-1}, x_k]\). For \(x_1 = x\) and \(x_2 = x_3 = \ldots = x_k = y\), denote \([x, y, \ldots, y]\) by \([x, y]_k\). A result of Herstein [1] and of Anan'in and Zyabko [2] asserts that if for any \(x\) and \(y\) in a ring \(R\), there exist positive integers \(m = m(x, y)\), \(n = n(x, y)\) such that \(x^m y^n = y^n x^m\), then the commutator ideal of \(R\) is nil. Recently, Herstein [3] proved that a ring \(R\) in which for any \(x, y, z \in R\) there exists positive integers \(m = m(x, y, z), n = n(x, y, z), \) and \(q = q(x, y, z)\) such that \([x^m, y^n, z^q] = 0\) must have nil commutator ideal. More recently Klein, Nada and Bell [4] raised the following conjecture which arises naturally from the above mentioned work.

CONJECTURE. Let \(k > 1\). If for each \(x, y \in R\), there exists positive integers \(m\) and \(n\) such that \([x^m, y^n]_k = 0\), then the commutator ideal of \(R\) is nil.

In [4], Klein, Nada and Bell proved the conjecture for rings with identity 1. Given the complexity of [1] and [3], it would appear that no proof of this conjecture is in sight. Our objective is to prove the conjecture for certain classes of rings and to generalize a result of Herstein in [3] and some results in [4] and [5].

A ring \(R\) is called periodic if for each \(x\) in \(R\), there exists distinct positive integers \(m\) and \(n\) for which \(x^m = x^n\). In preparation for the proofs of our main theorems, we start with the following lemma which is known [5] and we omit its proof.
LEMMA 1. If \( R \) is a periodic ring, then for each \( x \) in \( R \), there exists a positive integer \( k = k(x) \) such that \( x^k \) is idempotent.

2. MAIN RESULTS.

The following theorem shows that the conjecture is true for Artinian rings.

THEOREM 1. Let \( k > 1 \), and let \( R \) be an Artinian ring such that for each \( x, y \) in \( R \), there exists positive integers \( m \) and \( n \) such that \([x, y]_k = 0\). Then the commutator ideal of \( R \) is nil.

PROOF. To prove that the commutator ideal of \( R \) is nil it is enough to show that if \( R \) has no nonzero nil ideals then it is commutative. So we suppose that \( R \) has no nonzero nil ideals. Since \( R \) is Artinian, the Jacobson radical \( J \) of \( R \) is nilpotent. So \( J = 0 \), and hence \( R \) is semisimple Artinian. This implies that \( R \) has an identity element and now, \( R \) is commutative by Theorem 3 of [4].

Next, we prove Theorem 2 which shows that the conjecture is true for periodic rings. This result generalizes a result of Bell in [5].

THEOREM 2. Let \( k > 1 \) and let \( R \) be a periodic ring such that for each \( x, y \) in \( R \) there exists positive integers \( m \) and \( n \) such that \([x^m, y^n]_k = 0\). Then the commutator ideal of \( R \) is nil.

PROOF. If \( k = 2 \), then the result follows by the theorem in [1]. So assume \( k > 2 \) and let \( x \) be any element of \( R \) and let \( e \) be any idempotent of \( R \). By hypothesis, there exists integers \( m \) and \( n \) such that \([x^m, e^n]_k = 0\). This implies that \([x^m, e]_k = 0\), and hence

\[
[x^m, e]_{k-1}e = e[x^m, e]_k. 
\]

Multiplying by \( e \) from the right and using the fact that \( e[x^m, e]_k = e \) we obtain \([x^m, e]_{k-1}e = 0\). Hence \( 0 = ([x^m, e]_k - e - e[x^m, e]_k - e)e = [x^m, e]_{k-2}e \).

Continuing this way we get \([x^m, e] = 0\) which implies that \( x^m e = x^m e \). Similarly, we can get \( x^m e = x^m e \). This implies that

\[
x^m e = x^m e, x \in R, e \text{ any idempotent and } m = m(x, e). \tag{2.1}
\]

Let \( y \) be any element of \( R \). Since \( R \) is periodic, Lemma 1 implies that \( y^p \) is idempotent for some positive integer \( p = p(y) \). So (2.1) implies that for each \( x, y \) in \( R \) there exists positive integers \( m \) and \( p \) such that \( x^m y^p = y^p x^m \). Now, the result follows by the well-known theorem in [1] or [2].

THEOREM 3. Let \( k > 1 \). If \( R \) is a prime ring having a nonzero idempotent element such that for each \( x, y \) in \( R \) there exists positive integers \( m \) and \( n \) such that \([x^m, y^n]_k = 0\). Then \( R \) is commutative.

PROOF. The argument used in Theorem 2 to reach statement (2.1) in the proof shows that a ring satisfying the generalized commutator constraint \([x^m, y^n]_k = 0\) must have its idempotent elements in the center. For let \( e_1 \) and \( e_2 \) be idempotent elements in \( R \). (2.1) implies that \( e_1 e_2 = e_2 e_1 \) and hence the idempotents of \( R \) commute. This implies that the idempotents of \( R \) are central in \( R \) [6, Remark 2]. Let \( e \) be a nonzero
idempotent of $R$. Then $e$ is a nonzero central idempotent in the prime ring $R$. Hence $e$ is an identity element of $R$ since it can not be a zero divisor. Now $R$ is commutative by Theorem 3 of [4].

The proof of Theorem 4 below was done by Kezlan in the proof of his main theorem in [7]. So we omit its proof here.

**THEOREM 4.** Let $k > 1$. If $R$ is a prime ring with a nontrivial center such that for each $x, y$ in $R$ there exists positive integers $m$ and $n$ such that $[x^m, y^n]_k = 0$, then $R$ is commutative.

The following result generalizes Theorem 1 of [4].

**THEOREM 5.** Let $R$ be a ring and let $M$ be a fixed positive integer. Suppose that for each $x, y \in R$ there exist positive integers $m = m(x, y) < M$ and $n = n(x, y)$ such that $[x^m, y^n, y^n]$ belongs to the center of $R$. Then the commutator ideal of $R$ is nil.

**PROOF.** Again, we suppose that $R$ has no nil ideals and hence $R$ is a subdirect product of prime rings satisfying the above hypothesis of $R$. So we may assume that $R$ is prime. Let $Z$ be the center of $R$. If $Z = 0$, then for each $x, y \in R$, $[x^m, y^n, y^n] = 0$ where $m = m(x, y) < M$, and $n = n(x, y)$. This implies that $R$ is commutative by Theorem 4 above.

The following result generalizes Theorem 8 in [3].

**THEOREM 6.** Let $R$ be a ring in which, for each $x, y, z \in R$, there exists positive integers $m = m(x, y, z)$, $n = n(x, y, z)$, and $q = q(x, y, z)$ such that $[x^m, y^n, z^q] \in Z$ where $m$, $n$, $q$ are each functions of the variables $x$ and $y$. So $[[[x^m, y^n], y^q]], y^n] = 0$, which implies that $[[[x^m, y^n], y^q]] = 0$. Hence $R$ is commutative by Theorem 4 above.

**REMARK.** The result in Theorem 6 can be generalized as follows. Let $R$ be a ring such that for each $x, y, z \in R$, there exists positive integers $m = m(x, y, z)$, $n = n(x, y, z)$, and $q = q(x, y, z)$ such that $[x^m, y^n, z^q, r_1, r_2, ..., r_k] = 0$ for all elements $r_1, ..., r_k$ in $R$. Then the commutator ideal of $R$ is nil. This can be done by induction on $k$ and using the argument in Theorem 6. We omit the details of the proof.

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