A GENERALIZATION OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. Let $T_p$ be the class of analytic and $p$-valent functions which can be expressed in the form

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}, \quad |z| < 1.$$  

The subclasses $T_p^*(A,B,a)$ and $C_p(A,B,a)$ of $T_p$ have been considered. Sharp results concerning coefficient estimates, distortion and covering theorems are obtained. The radius of convexity for the class $T_p^*(A,B,a)$ is determined. It is further proved that the classes $T_p^*(A,B,a)$ and $C_p(A,B,a)$ are closed under arithmetic mean and convex linear combinations.

KEY WORDS AND PHRASES. $p$-valent, Analytic, Radius of Convexity.

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1. INTRODUCTION.

Let $S_p(p > 1)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$$

which are analytic and $p$-valent in the unit disc $U = \{z: |z| < 1\}$. A function $f$ is said to be subordinate to a function $F(f < F)$ if there exists an analytic function $\phi(z)$ with $|\phi(z)| < |z|$, $z \in U$, such that $f = f \cdot \phi$.

For $A, B$ fixed, $-1 < A < B < 1$, and $0 < a < p$, we say that $f \in S_p^*(A,B,a)$ if and only if

$$\frac{zf'(z)}{f(z)} < \frac{p + [pB+(A-B)(p-a)]z}{1 + Bz}, \quad z \in U,$$
or equivalently \( f \in S_p^*(A,B,\alpha) \) if and only if
\[
\frac{zf'(z) - \alpha p(f(z))}{f(z)} - \frac{[pB+(A-B)(\alpha-\alpha)]}{f(z)} < 1, \ z \in U.
\]
Further \( f \) is said to belong to the class \( K_p^*(A,B,\alpha) \) if and only if
\[
\frac{zf'(z) - \alpha p(f(z))}{f(z)} - \frac{[pB+(A-B)(\alpha-\alpha)]}{f(z)} \leq 1, \ z \in U.
\]

Let \( T_p \) denote the subclass of \( S_p^* \) consisting of functions analytic and \( p \)-valent which can be expressed in the form
\[
f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n}, \text{ and we set}
\]
\[
\]

Silverman [3], Gupta and Jain [2] and Silverman and Silvia [4,5] have studied certain subclasses of univalent functions with negative coefficients. Also Goel and Sohi [1] have studied certain subclasses of multivalent functions with negative coefficients. In this paper we obtain coefficient estimates, distortion and covering theorems for the classes \( T_p^*(A,B,\alpha) \) and \( C_p(A,B,\alpha) \). We also determine the radius of convexity for the class \( T_p^*(A,B,\alpha) \). It is further shown that the classes \( T_p^*(A,B,\alpha) \) and \( C_p(A,B,\alpha) \) are closed under arithmetic mean and convex linear combinations. By taking \( \alpha = 0 \), we get results due to Goel and Sohi [1] and by assigning specific values to \( A \) and \( B \) and taking \( p = 1 \), we get results due to Silverman [3] and Gupta and Jain [2].

2. COEFFICIENT INEQUALITIES.

**THEOREM 1.** A function \( f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \) is in \( T_p^*(A,B,\alpha) \) if and only if
\[
\left( \sum_{n=1}^{\infty} [(1+B)n+(B-A)(\alpha-\alpha)] |a_{p+n}| \right) z^p \leq (B-A)(\alpha-\alpha).
\]

The result is sharp.

**PROOF.** Let \( \|z\| = 1 \), then
\[
|zf'(z) - \alpha p(f(z))| - B (zf'(z) - [pB+(A-B)(\alpha-\alpha)] f(z)) |
\]
\[
= \left| \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \right| - (B-A)(\alpha-\alpha) z^p
\]
\[
- \sum_{n=1}^{\infty} [nB + (B-A)(\alpha-\alpha)] |a_{p+n}| z^{p+n} |
\]
\[
\leq \sum_{n=1}^{\infty} [(1+B)n+(B-A)(\alpha-\alpha)] |a_{p+n}| - (B-A)(\alpha-\alpha) < 0.
\]

Hence by the principle of maximum modulus \( f(z) \in T_p^*(A,B,\alpha) \).

Conversely, suppose that
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\[
\frac{zf'(z)}{f(z)} - \rho = \frac{zf'(z)}{f(z)} - \left[ PB + (A-B)(p-a) \right]
\]

\[
= \left| \sum_{n=1}^{\infty} n|a_{p+n}|z^{p+n} \right| \left( B-A \right)(p-a)z^p - \sum_{n=1}^{\infty} \left[ nB+(B-A)(p-a) \right]|a_{p+n}|z^{p+n}
\]

\[
< 1, \quad z \in U.
\]

Since \( |\text{Re} \, z| < |z| \) for all \( z \), we have

\[
\text{Re} \left( \sum_{n=1}^{\infty} n|a_{p+n}|z^{p+n} \right) \left( B-A \right)(p-a)z^p - \sum_{n=1}^{\infty} \left[ nB+(B-A)(p-a) \right]|a_{p+n}|z^{p+n}
\]

\[
< 1. \quad (2.2)
\]

Choose values of \( z \) on the real axis so that \( \frac{zf'(z)}{f(z)} \) is real. Upon clearing the denominator in (2.2) and letting \( z \) through real values, we obtain

\[
\sum_{n=1}^{\infty} n|a_{p+n}| < \left[ (B-A)(p-a) - \sum_{n=1}^{\infty} \left[ nB+(B-A)(p-a) \right]|a_{p+n}| \right]
\]

which implies that

\[
\sum_{n=1}^{\infty} \left[ (1+B)n + (B-A)(p-a) \right]|a_{p+n}| < (B-A)(p-a).
\]

The function

\[
f(z) = z^p - \sum_{n=1}^{\infty} (B-A)(p-a) \left( (1+B)n + (B-A)(p-a) \right)z^{p+n}
\]

is an extremal function.

COROLLARY 1. If \( f \in T^*_p(A,B,a) \) then

\[
\sum_{n=1}^{\infty} (1+B)n + (B-A)(p-a)\left| a_{p+n} \right| < (B-A)(p-a),
\]

with equality only for functions of the form

\[
f(z) = z^p - \frac{(B-A)(p-a)}{(1+B)n + (B-A)(p-a)}z^{p+n}.
\]

COROLLARY 2. A function \( f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \) is in \( C_p(A,B,a) \) if and only if

\[
\sum_{n=1}^{\infty} \left( \frac{ntp}{p} \right) (1+B)n + (B-A)(p-a)\left| a_{p+n} \right| < (B-A)(p-a).
\]

PROOF. It is well known that \( f \in C_p(A,B,a) \) if and only if \( \frac{zf'(z)}{f(z)} \in T^*_p(A,B,a) \).

Since

\[
\frac{zf'(z)}{f(z)} = z^p - \sum_{n=1}^{\infty} \frac{(ntp)}{p} \left| a_{p+n} \right|z^{p+n}
\]

we may replace \( |a_{p+n}| \) with \( \frac{(ntp)}{p} |a_{p+n}| \) in Theorem 1.
3. REPRESENTATION FORMULA.

THEOREM 2. A function \( f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}z^{n+p} \) is in \( T^p(A, B, a) \) if and only if

\[
f(z) = z^p \exp \left\{ \int^z_0 \frac{\phi(t)}{1-Bt} \, dt \right\},
\]

(3.1)

where \( \phi(z) \) is analytic in \( U \) and satisfies \( |\phi(z)| < 1, \, z \in U \).

PROOF. Let \( f(z) \in T^p(A, B, a) \), then

\[
\frac{zf'(z)}{f(z)} = \frac{Bzf'(z) - (pB+(A-B)(p-a))}{zf(z)} < 1, \, z \in U.
\]

Since the absolute value vanishes for \( z = 0 \), we have

\[
\frac{zf'(z)}{f(z)} = h(z)
\]

(3.2)

where \( h(z) \) is analytic in \( U \) and \( |h(z)| < 1 \) for \( z \in U \). Integrating (3.2) with

\[
f(z) = z^p \exp \left\{ \int^z_0 \frac{\phi(t)}{1-Bt} \, dt \right\}.
\]

The converse is obtained by differentiating (3.1).

4. DISTORTION AND COVERING THEOREMS FOR \( T^p(A, B, a) \) and \( C_p(A, B, a) \).

THEOREM 3. If \( f(z) \in T^p(A, B, a) \), then

\[
r^p - \frac{(B-A)(p-a)}{1+B+(B-A)(p-a)} r^{p+1} < |f(z)| < \frac{r^p + (B-A)(p-a)}{1+B+(B-A)(p-a)} r^{p+1} \quad (|z| = r),
\]

(4.1)

with equality for \( f(z) = z^p - \frac{(B-A)(p-a)}{1+B+(B-A)(p-a)} z^{p+1}(z+r) \).

PROOF. From Theorem 1, we have

\[
\sum_{n=1}^{\infty} |a_{p+n}| [1+B+(B-A)(p-a)] < \sum_{n=1}^{\infty} [(1+B)(n+B)(p-a)] |a_{p+n}| < (B-A)(p-a).
\]

This implies that

\[
\sum_{n=1}^{\infty} |a_{p+n}| < \frac{(B-A)(p-a)}{1+B+(B-A)(p-a)}.
\]

(4.2)

Thus

\[
|f(z)| < |z|^p + \sum_{n=1}^{\infty} |a_{p+n}| |z|^{p+n}
\]

\[
< r^p(1 + r \sum_{n=1}^{\infty} |a_{p+n}|)
\]
Similarly,

\[|f(z)| > |z|^p - \sum_{n=1}^{\infty} |a_{p+n}| \cdot |z|^{p+n} \]

\[> r^p (1 - r \sum_{n=1}^{\infty} |a_{p+n}|) \]

\[> r^p - \frac{(B-A)(p-a)}{1+B+(B-A)(p-a)} r^{p+1}. \]

**COROLLARY 3.** If \( f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \in C_p(A,B,a) \), then

\[r^p - \frac{(B-A)(p-a)}{1+B+(B-A)(p-a)} r^{p+1} < |f(z)| < \]

\[r^p + \frac{(B-A)(p-a)(p+1)}{1+B+(B-A)(p-a)} r^{p+1} (|z| = r), \]

with equality for

\[f(z) = z^p - \frac{(B-A)(p-a)(p+1)}{p[1+B+(B-A)(p-a)]} z^{p+1} (z = \pm r). \]

**THEOREM 4.** The disc \(|z| < 1\) is mapped onto a domain that contains the disc \(|w| < \frac{1 + B}{1+B+(B-A)(p-a)}\) by any \( f \in T^*_p(A,B,a) \), and onto a domain that contains the disc \(|w| < \frac{p[p+B-(A-B)(p-a)]}{p[p+1+B+(B-A)(p-a)]}\) by any \( f \in C_p(A,B,a) \).

The theorem is sharp, with extremal functions

\[z^p - \frac{(B-A)(p-a)}{1+B+(B-A)(p-a)} z^{p+1} \in T^*_p(A,B,a) \text{ and} \]

\[z^p - \frac{(B-A)(p-a)(p+1)}{p[1+B+(B-A)(p-a)]} z^{p+1} \in C_p(A,B,a). \]

**PROOF.** The results follow upon letting \( r + 1 \) in Theorem 3 and Corollary 3.

**THEOREM 5.** If \( f \in T^*_p(A,B,a) \), then

\[pr^{p-1} - \frac{(p+1)(B-A)(p-a)}{1+B+(B-A)(p-a)} r^p < |f'(z)| < \]
\[ p r^{p-1} + \frac{(p+1)(B-A)(p-a)}{1+B+(B-A)(p-a)} r^p \left( |z| = r \right). \]

Equality holds for

\[ f(z) = z^p - \frac{(B-A)(p-a)}{1+B+(B-A)(p-a)} z^{p+1} \left( z = \pm r \right). \]

**PROOF.** We have

\[
|f'(z)| < pr^{p-1} + \sum_{n=1}^{\infty} (p+n) |a_{p+n}| r^{p+n-1} \\
< pr^{p-1} + r^p \sum_{n=1}^{\infty} (p+n) |a_{p+n}| \\
= r^{p-1} \left[ p+r \sum_{n=1}^{\infty} (p+n) |a_{p+n}| \right]. \tag{4.3}
\]

In view of Theorem 1,

\[
\sum_{n=1}^{\infty} (1+B) \left[ n+p- \frac{p(1+B)+(A-B)(p-a)}{1+B} \right] |a_{n+p}| \leq (B-A)(p-a) \]

or

\[
\sum_{n=1}^{\infty} (1+B)(n+p) |a_{n+p}| < (B-A)(p-a) + \]

\[
[p(1+B)+(A-B)(p-a)] \sum_{n=1}^{\infty} |a_{n+p}| \tag{4.4}
\]

(4.4) with the help of (4.2) implies that

\[
\sum_{n=1}^{\infty} (n+p) |a_{n+p}| < \frac{(p+1)(B-A)(p-a)}{1+B+(B-A)(p-a)} . \tag{4.5}
\]

A substitution of (4.5) into (4.3) yields the right-hand inequality.

On the other hand

\[
|f'(z)| > r^{p-1} \left[ p-r \sum_{n=1}^{\infty} (p+n) |a_{p+n}| \right] \\
> pr^{p-1} - \frac{(p+1)(B-A)(p-a)}{1+B+(B-A)(p-a)} r^p .
\]

This completes the proof.
COROLLARY 4. If \( f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p} \in C_p(A,B,a) \), then

\[
pr^{p-1} - \frac{(B-A)(p-a)(p+1)^2}{p[1+B+(B-A)(p-a)]} r^p < |f'(z)| < 
\]

\[
pr^{p-1} + \frac{(B-A)(p-a)(p+1)^2}{p[1+B+(B-A)(p-a)]} r^p \quad (|z| = r).
\]

Equality holds for \( f(z) = z^p - \frac{(B-A)(p-a)(p+1)}{p[1+B+(B-A)(p-a)]} z^{p+1} \).

5. RADIUS OF CONVEXITY FOR THE CLASS \( T^*_p (A,B,a) \).

THEOREM 6. If \( f(z) \in T^*_p (A,B,a) \), then \( f(z) \) is \( p \)-valently convex in the disc \( |z| < R_p \).

\[
\left| f(z) \right| < R_p = \inf \left\{ \frac{(1+B)n+(B-A)(p-a)}{(B-A)(p-a)} \frac{(B-A)(p-a)}{n+p} \right\} (n=1,2,...).
\] (5.1)

The result is sharp, with the extremal function

\[
f(z) = z^p - \frac{(B-A)(p-a)}{(1+B)n+(B-A)(p-a)} z^{p+n}.
\]

PROOF. It is sufficient to show that \( |1 + \frac{zf''(z)}{f'(z)}| - p < p \) for \( |z| < R_p \).

We have

\[
\left| 1 + \frac{zf''(z)}{f'(z)} \right| - p = \left| -\sum_{n=1}^{\infty} \frac{n(n+p)}{p-n+1} \frac{|a_{n+p}| z^n}{|a_{n+p}| |z|^n} \right| < p
\]

Thus

\[
\left| 1 + \frac{zf''(z)}{f'(z)} \right| - p < p \text{ if}
\]

\[
\sum_{n=1}^{\infty} (n+p)^2 |a_{n+p}| |z|^n < p^2
\]

or

\[
\sum_{n=1}^{\infty} \frac{(n+p)^2}{p} |a_{n+p}| |z|^n < 1.
\] (5.2)

According to Theorem 1, \( \sum_{n=1}^{\infty} \frac{(1+B)n+(B-A)(p-a)}{(B-A)(p-a)} |a_{n+p}| < 1 \).
Hence (5.2) will be true if
\[
\left( \frac{n+p}{p} \right)^2 |z|^n < \frac{(1+B)n+(B-A)(p-a)}{(B-A)(p-a)}
\]
or if
\[
|z| < \left[ \frac{(1+B)n+(B-A)(p-a)}{(B-A)(p-a)} \right], \left( \frac{p}{n+p} \right)^2 n^n (n=1,2,\ldots).
\] (5.3)
The theorem follows easily from (5.3).

6. CLOSURE THEOREMS.

In this section we shall prove that the classes \( T^*_p(A,B,a) \) and \( C^*_p(A,B,a) \) are closed under convex linear combinations.

THEOREM 7. If \( f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \) and \( g(z) = z^p - \sum_{n=1}^{\infty} b_{n+p} z^{n+p} \) are in \( T^*_p(A,B,a) \), then \( h(z) = z^p - \frac{1}{2} \sum_{n=1}^{\infty} a_{n+p} + b_{n+p} z^{n+p} \) is also in \( T^*_p(A,B,a) \).

PROOF. Since \( f(z) \) and \( g(z) \) are in \( T^*_p(A,B,a) \), we have
\[
\sum_{n=1}^{\infty} [(1+B)n+(B-A)(p-a)] |a_{n+p}| z^{n+p} < (B-A)(p-a) \] (6.1)
and
\[
\sum_{n=1}^{\infty} [(1+B)n+(B-A)(p-a)] |b_{n+p}| z^{n+p} < (B-A)(p-a). \] (6.2)
From (6.1) and (6.2) we get
\[
\frac{1}{2} \sum_{n=1}^{\infty} [(1+B)n+(B-A)(p-a)] |a_{n+p} + b_{n+p}| z^{n+p} < (B-A)(p-a)
\]
which implies that \( h(z) \in T^*_p(A,B,a) \).

The following theorem can be proven similarly.

THEOREM 8. If \( f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \) and \( g(z) = z^p - \sum_{n=1}^{\infty} b_{n+p} z^{n+p} \) are in \( C^*_p(A,B,a) \), then \( h(z) = z^p - \frac{1}{2} \sum_{n=1}^{\infty} a_{n+p} + b_{n+p} z^{n+p} \) is also in \( C^*_p(A,B,a) \).

THEOREM 9. Let \( f_p(z) = z^p, f_{n+p}(z) = z^p - \frac{(B-A)(p-a)}{(1+B)n+(B-A)(p-a)} z^{n+p} \) (n=1,2,3,\ldots).

Then \( f \in T^*_p(A,B,a) \) if and only if it can be expressed in the form
\[
f(z) = \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z) \text{ where } \lambda_{n+p} \geq 0 \text{ and } \sum_{n=0}^{\infty} \lambda_{n+p} = 1.
\]

PROOF. Suppose \( f(z) = \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z) = z^p \sum_{n=0}^{\infty} \frac{(B-A)(p-a)}{(1+B)n+(B-A)(p-a)} \lambda_{n+p} z^{n+p}. \)
Then
\[
\sum_{n=1}^{\infty} \left[ \lambda^{n+p} \frac{(1+B)n+(B-A)(p-a)}{(B-A)(p-a)} \right. \cdot \left. \frac{(B-A)(p-a)}{(I+B)n+(B-A)(p-a)} \right] \\
= \sum_{n=1}^{\infty} \lambda^{n+p} < 1 - \lambda < 1.
\]

So by Theorem 1, \( f(z) \in \mathcal{T}_p^{\ast}(A,B,a) \).

Conversely suppose \( f(z) \in \mathcal{T}_p^{\ast}(A,B,a) \). Then
\[
|a_{n+p}| < \frac{(B-A)(p-a)}{(I+B)n+(B-A)(p-a)}.
\]

Setting \( \lambda^{n+p} = \frac{(I+B)n+(B-A)(p-a)}{(B-A)(p-a)} \left| a_{n+p} \right| (n=1,2,\ldots) \),

and
\[
\lambda = 1 - \sum_{n=1}^{\infty} \lambda^{n+p},
\]

we have
\[
f(z) = \sum_{n=0}^{\infty} \lambda^{n+p} f_{n+p}(z).
\]

This completes the proof of the theorem.

REMARKS. (1) Putting \( a=0 \) in the above theorems we get the results obtained by R.M. Goel and N.S. Sohi [1].

(2) Putting \( p=1 \) and taking \( A=-\beta, B=\beta, \) where \( 0 < \beta < 1 \), in the above theorems we get the results obtained by Gupta and Jain [2].

(3) Putting \( p=1 \) and taking \( A=-1, B=1 \) in the above theorems we get the results obtained by Silverman [3].

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REFERENCES


