ON FIRST-ORDER DIFFERENTIAL OPERATORS WITH BOHR-NEUGEBAUER TYPE PROPERTY

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ABSTRACT. We consider a differential equation $\frac{d}{dt} u(t) - Bu(t) = f(t)$, where the functions u and f map the real line into a Banach space X and $B: X \rightarrow X$ is a bounded linear operator. Assuming that any Stepanov-bounded solution u is Stepanov almost-periodic when f is Bochner almost-periodic, we establish that any Stepanov-bounded solution u is Bochner almost-periodic when f is Stepanov almost-periodic. Some examples are given in which the operator $\frac{d}{dt} - B$ is shown to satisfy our assumption.

KEY WORDS AND PHRASES. Bounded linear operator, differential operator, Bohr-Neugebauer property, Bochner (Stepanov) almost-periodic function. 1980 AMS SUBJECT CLASSIFICATION CODE. 34Gxx, 34G10, 34C27.

1. INTRODUCTION.

Suppose X is a Banach space and J is the interval $-\infty < t < \infty$. A function $f \in L^p_{loc}(J; X)$ with $1 \leq p < \infty$ is said to be Stepanov-bounded or S^p -bounded on J if

$$\|f\|_{S^p} = \sup_{t \in J} \left[\int_t^{t+1} \|f(s)\|^p ds \right]^{1/p} < \infty. \quad (1.1)$$

Our first result is as follows.

THEOREM 1. Suppose $f: J \rightarrow X$ is a continuously differentiable S^1 -bounded function, and f' is an S^p -bounded function with $1 \leq p < \infty$. Then, (a) if $p = 1$, f is bounded on J , and (b) if $p > 1$, f is bounded and uniformly continuous on J .

2. PROOF OF THEOREM 1.

(a) $p = 1$. For an arbitrary but fixed $t \in J$, there exists at least one point $\tau_t \in [t-1, t]$ such that

$$\|f(\tau_t)\| = \inf_{t-1 \leq s \leq t} \|f(s)\|. \quad (2.1)$$

Consequently, we have

$$\|f(\tau_t)\| \leq \int_{t-1}^t \|f(s)\| ds \leq \|f\|_{S^1}, \text{ by (1.1).} \quad (2.2)$$

Hence, from the S^1 -boundedness of f' , we obtain

$$\begin{aligned} \|f(t)\| &= \|f(\tau_t) + \int_{\tau_t}^t f'(s) ds\| \\ &\leq \|f(\tau_t)\| + \int_{\tau_t}^t \|f'(s)\| ds \\ &\leq \|f\|_{S^1} + \|f'\|_{S^1}. \end{aligned} \quad (2.3)$$

(b) $p > 1$. By Hölder's inequality, the S^p -boundedness of f' implies the S^1 -boundedness of f' . Hence, as shown above, f is bounded on J .

Moreover, for $0 < t_2 - t_1 < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have, again by Hölder's inequality,

$$\begin{aligned} \|f(t_2) - f(t_1)\| &= \left\| \int_{t_1}^{t_2} f'(s) ds \right\| \\ &\leq (t_2 - t_1)^{1/q} \left[\int_{t_1}^{t_2} \|f'(s)\|^p ds \right]^{1/p} \\ &\leq (t_2 - t_1)^{1/q} \left[\int_{t_1}^{t_1+1} \|f'(s)\|^p ds \right]^{1/p} \\ &\leq (t_2 - t_1)^{1/q} \|f'\|_{S^p}. \end{aligned} \quad (2.4)$$

Therefore f is uniformly continuous on J , completing the proof of the theorem.

REMARK. If $f: J \rightarrow X$ is a continuously differentiable S^1 -almost periodic function, with f' being S^p -bounded on J ($1 < p < \infty$), then f is (uniformly) almost-periodic from J to X (see pp. 3 and 77, Amerio-Prouse [1] for the definitions of (uniform) almost-periodicity and S^p -almost periodicity).

PROOF. By Theorem 1, f is uniformly continuous on J . Hence, by Theorem 7, p. 78, Amerio-Prouse [1], f is (uniformly) almost-periodic from J to X .

3. MAIN RESULT.

Let B be a bounded linear operator on a Banach space X into itself. Then the differential operator $\frac{d}{dt} - B$ is said to have Bohr-Neugebauer property if, for any (uniformly) almost-periodic X -valued function f , any bounded (on J) solution of the equation

$$\frac{d}{dt} u(t) - Bu(t) = f(t) \text{ on } J \quad (3.1)$$

is (uniformly) almost-periodic.

Our result is as follows.

THEOREM 2. In a Banach space X , let the differential operator $\frac{d}{dt} - B$ be such that, for any (uniformly) almost-periodic X -valued function f , any S^1 -bounded solution of the equation (3.1) is S^1 -almost periodic. Then, for any S^1 -almost periodic continuous X -valued function g , any S^1 -bounded solution $u: J \rightarrow X$ of the equation

$$\frac{d}{dt} u(t) - Bu(t) = g(t) \text{ on } J \quad (3.2)$$

is (uniformly) almost-periodic.

PROOF. Since g is S^1 -almost periodic from J to X , it is S^1 -bounded on J . Consequently, $u' = Bu + g$ is S^1 -bounded on J . Hence, by Theorem 1, u is bounded on J .

Now consider a sequence $\{\phi_n(t)\}_{n=1}^\infty$ of non-negative continuous functions on J such that

$$\phi_n(t) = 0 \quad \text{for } |t| \geq n^{-1}, \quad \int_{-n}^{n-1} \phi_n(t) dt = 1. \quad (3.3)$$

The convolution of u and ϕ_n is defined by

$$(u * \phi_n)(t) = \int_J u(t-s) \phi_n(s) ds = \int_J u(s) \phi_n(t-s) ds. \quad (3.4)$$

Then, by (3.2), we have

$$\frac{d}{dt} (u * \phi_n)(t) - B(u * \phi_n)(t) = (g * \phi_n)(t) \quad \text{on } J. \quad (3.5)$$

We note that

$$\sup_{t \in J} \|(u * \phi_n)(t)\| \leq \sup_{t \in J} \|u(t)\|. \quad (3.6)$$

Further, we can show that $g * \phi_n$ is (uniformly) almost-periodic from J to X (see the proof of Theorem 7, p. 78, Amerio-Prouse [1]).

Therefore, by our assumption on the operator $\frac{d}{dt} - B$, $(u * \phi_n)(t)$ is S^1 -almost periodic for all $n = 1, 2, \dots$.

By (3.2), we have the representation

$$u(t) = u(0) + \int_0^t Bu(s) ds + \int_0^t g(s) ds \quad \text{on } J. \quad (3.7)$$

If $t_2 > t_1$, then

$$\left\| \int_{t_1}^{t_2} Bu(s) ds \right\| \leq \|B\| \cdot \sup_{t \in J} \|u(t)\| \cdot (t_2 - t_1). \quad (3.8)$$

Hence $\int_0^t Bu(s) ds$ is uniformly continuous on J . Also, by Theorem 8, p. 79, Amerio-Prouse [1], $\int_0^t g(s) ds$ is uniformly continuous on J . Consequently, u is uniformly continuous on J .

Similarly, from (3.5), it follows that $u * \phi_n$ is uniformly continuous on J . So, by Theorem 7, p. 78, Amerio-Prouse [1], $u * \phi_n$ is (uniformly) almost-periodic for all $n = 1, 2, \dots$.

Now, by the uniform continuity of u on J , the sequence of convolutions $(u * \phi_n)(t)$ converges to $u(t)$ uniformly on J . Hence u is (uniformly) almost-periodic from J to X , which completes the proof of the theorem.

4. NOTES.

(i) Suppose X is a Hilbert space and B is a self-adjoint bounded linear operator on X into itself. Then we know that the operator $\frac{d}{dt} - B$ has Bohr-Neugebauer property (see Zaidman [4]). Given an (uniformly) almost-periodic X -valued function f , suppose that u is an S^1 -bounded solution of the equation (3.1). If we replace g by f in the proof of our Theorem 2, then, by the Bohr-Neugebauer property of the operator $\frac{d}{dt} - B$, it follows that u is (uniformly) almost-periodic from J to X . Thus the operator $\frac{d}{dt} - B$ satisfies the hypothesis of Theorem 2.

(ii) Now suppose X is a separable Hilbert space and B is a completely continuous normal operator on X into itself. Then, by Theorem 1 of Cooke [3], the operator $\frac{d}{dt} - B$ has Bohr-Neugebauer property. Consequently, the operator $\frac{d}{dt} - B$ satisfies the assumption of Theorem 2.

(iii) Finally, suppose X is a reflexive space and $B = 0$. Then the operator $\frac{d}{dt}$ has Bohr-Neugebauer property (see Amerio-Prouse [1], p. 55 and Authors' Remark on p. 82). Hence the operator $\frac{d}{dt}$ satisfies the assumption of Theorem 2.

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