SUBLINEAR FUNCTIONALS ERGODICITY AND FINITE INVARIANT MEASURES

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ABSTRACT. By introducing a sublinear functional involving infinite matrices, we establish its connection with ergodicity and measure preserving transformation. Further, we characterize the existence of a finite invariant measure by means of a condition involving the above sublinear functional.

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1. INTRODUCTION AND DEFINITIONS.

Let \( \ell_\infty \) be the set of all real bounded sequence \( \{x_n\} \), normed by \( \|x\| = \sup_{n \geq 0} |x_n| \). Linear functional \( \phi \) on \( \ell_\infty \) are called Banach limit [1] satisfying the conditions,

i) \( \phi(x_n) \geq 0 \), if \( x_n \geq 0 \), \( n = 0,1,2 \ldots \).

ii) \( \phi(x_{n+1}) = \phi(x_n) \).

iii) \( \lim_{n \to \infty} x_n \leq \phi(x_n) \leq \lim_{n \to \infty} x_n \).

If there is a number for all Banach limits \( \phi \), the sequence \( x = \{x_n\} \) is called almost convergent and we write; \( F - \lim_{n \to \infty} x_n = s \). It is shown by Lorentz [2] that a sequence \( \{x_n\} \) is almost convergent with F-limit \( s \), if and only if

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=i}^{i+n-1} x_k = s.
\]

uniformly in \( i \).
Let, $A = (a_{n,k}^{(i)})$ be a sequence of real or complex matrices for each $i = 0, 1, 2\ldots$ such that $a_{n,k}^{(i)} = 0$, if any $n, k, i$, is a negative integer. The sequence $\{x_n\}$ is called $A$ summable to $s$ if

$$\lim_{n \to \infty} \frac{1}{k} \sum_{k=0}^{n} a_{n,k}^{(i)} x_k = s$$

uniformly in $i$ and in this case we write:

$$A \lim_{n \to \infty} x_n = s, \text{ or } x_n \to s(A).$$

In the case $a_{n,k}^{(i)} = 1/n+i$ $(i \leq k \leq i+n)$ and 0 otherwise, $(A)$ reduces to the method $(F)$. If $A = A = a_{n,k}$, then we obtain the usual summability method $(A)$. It is significant to note that there does not exist any regular method $(A)$ equivalent to method $(F)$ (See Lorentz [2], Theorem 11 and 12). In the case $a_{n,k}^{(i)} = \frac{1}{n+i}$, then $(A)$ reduces to the almost summability method introduced by King [3].

The method $(A)$ is called conservative, if $x \to s \Rightarrow x \to s^1$ $(A)$, regular, if $s = s^1$. The following characterization of regular matrices is due to Stieglitz [4]. The method $(A)$ is called regular if and only if the following conditions hold:

$$\sup_{i \geq 0, n \geq m} \left| a_{n,k}^{(i)} \right| < \infty, \text{ for all } n \text{ and } i \geq 0,$$

and there exists an integer $m$ such that

$$\sup_{i \geq 0, n \geq m} \left| a_{n,k}^{(i)} \right| < \infty$$

$$\lim_{n \to \infty} \frac{1}{k} \sum_{k=0}^{n} a_{n,k}^{(i)} = 1, \text{ uniformly in } i,$$

$$\lim_{n \to \infty} a_{n,k}^{(i)} = 0, \text{ for fixed } k, \text{ uniformly in } i.$$  

We write

$$||A|| = \sup_{i \geq 0, n \geq 0} \frac{1}{k} \sum_{k=0}^{n} \left| a_{n,k}^{(i)} \right|$$

The matrix $A$ is called translative, if

$$\lim_{n \to \infty} \frac{1}{k} \sum_{k=0}^{n} \left| d_{n,k}^{(i)} \right| = 0$$

uniformly in $i$, where

$$d_{n,k}^{(i)} = (a_{n,k-1}^{(i)} - a_{n,k}^{(i)})$$

The matrix $A$ is called positive, if

$$a_{n,k}^{(i)} \geq 0, \forall n, k, i$$

For real $\lambda$ we write,

$$\lambda^+ = \max(\lambda, 0), \quad \lambda^- = \max(-\lambda, 0).$$

The matrix $A$ is called almost positive, if

$$\lim_{n \to \infty} \frac{1}{k} \sum_{k=0}^{n} \left| a_{n,k}^{(i)} \right| = 0, \text{ uniformly in } i.$$
Let \((x, F, m)\) be a finite measure space and let \(T : X \to X\) be a measurable transformation. (This is assumed throughout). The measure \(m\) is called null invariant if \(m(A) = 0\) for all \(n\) and \(A \in F\). It is conservative, if \(A \cap T^{-n}A = \emptyset\) for all \(n\) and \(A \in F\). A measure \(\mu\) is called equivalent to measure \(m\), if \(m(A) \leftrightarrow \mu(A) = 0\) for all \(A \in F\). The transformation \(T\) is called measure preserving or invariant, if \(m(A) = m(T^{-1}A), A \in F\). It is called ergodic if \(T^{-1}A = A \Rightarrow m(A) = 0\) or \(m(X/A) = 0\). The set \(A \in F\) is called invariant, if \(A \cap T^{-n}A = A \in F\) are mutually disjoint. It is called weakly wandering, if there is an increasing sequence of positive integers \(\{r_k, k = 1, 2, 3, \ldots\}\) such that \(A, T^{-r_1}A, T^{-r_2}A \ldots \) are mutually disjoint. A measure \(\mu\) is called \(m\)-continuous if \(m(A) < \epsilon\) for each \(\epsilon > 0\) there exists a \(\delta > 0\) such that \(m(A) < \delta \Rightarrow \mu(A) < \epsilon\). A sequence of measures \(\{\mu_n\}\) is called uniformly \(m\)-continuous if for each \(\epsilon > 0\), there exists a \(\delta > 0\) such that \(m(A) < \delta \Rightarrow \mu_n(A) < \epsilon\) for all \(n\).

Write:

\[
t(x) = \lim_{n \to \infty} \sup_{i} \frac{1}{n} \sum_{k=1}^{i+n-1} x_k, \quad x_k \in \ell_\infty.
\]

Let \(\{l_\infty, t\}\) denote the set of linear functionals \(t\), such that \(\phi(x) \leq t(x)\). It is known (see Sucheston [5] Das and Misra [6]) that \(\{l_\infty, t\}\) is the set of all Banach limits on \(l_\infty\) and \(\phi \in \{l_\infty, t\}\) is unique if and only if \(\phi(x) = -\phi(-x)\) and this happens when

\[
\frac{1}{n} \sum_{k=1}^{i+n-1} x_k \to 0
\]

as \(n \to \infty\), uniformly in \(i\). Lorentz [2] calls all such sequences as almost convergent sequences. Let \(A\) be real and such that \(\|A\| < \infty\). Then we define, \(R : l_\infty \to l_\infty\) by

\[
R(x) = \lim_{n \to \infty} \sup_{i} \sum_{k=1}^{n} a_{n,k} x_k.
\]

Since, for all \(x \in l_\infty\)

\[
|R(x)| \leq \|x\| \|A\| < \infty,
\]

\(R\) is finite valued. It is easy to see that it is a sublinear functional on \(l_\infty\). By Hahn-Banach theorem there exists a linear functional \(\phi\) on \(l_\infty\) such that

\[
-R(-x) \leq \phi(x) \leq R(x), \quad x \in l_\infty
\]

Let \(\{l_\infty, R\}\) be the set of all linear functional \(\phi\) satisfying (1.13). It is easily seen that \(\phi\) is unique if and only if

\[
-R(-x) = R(x)
\]

and this happens if and only if

\[
\sum_{k=1}^{n} a_{n,k} x_k \to \text{a limit}
\]

as \(n \to \infty\), uniformly in \(i\).

We now state a lemma.

**Lemma 1.** Let \(x \in l_\infty\), then

(a) \(\lim n x_n \leq R(x) \leq \lim x_n\)
if and only if \( A \) is real, regular and almost positive.

(b) \(-t(-x) \leq -R(-x) \leq R(x) \leq t(x)\)

if and only if \( A \) is regular, almost positive and translative.

(c) If \( x \) is almost convergent to \( s \), then

\[
\lim_{n \to \infty} \sup_{k \geq 0} a_{n,k}^{(i)} x_k = s, \text{ uniformly in } i.
\]

2. ERGODICITY.

In this section, we establish that the ergodicity and invariance can be established
in terms of summability of a particular sequence and thus generalizes a result of
(Sucheston [5], Theorem 3) involving almost convergence.

We now examine the following conditions:

(I) For some \( \phi \in \{1, R\} \)

\( \phi(m(T^{-n} B \cap C)) = m(B) \cap m(C), \quad n = 0, 1, 2 \ldots \) .

(II) \( \lim_{n \to \infty} a_{n,k}^{(i)} m(T^{-n} B \cap C) = m(B) \cap m(C) \)

uniformly in \( i \), \( \forall B, C \in F \).

(III) \( T \) is ergodic and measure preserving.

THEOREM 1. Let \((X, F, m)\) be a finite measure space and let \( ||A|| < \infty \). Then

(a) (II) \( \Rightarrow \) (I)

(b) (i) (I) \( \Rightarrow \) T is ergodic

(ii) If \( A \) is translative. Then

(I) \( \Rightarrow \) (III)

(c) If \( A \) is regular, almost positive, and translative, then

(I) \( \Leftrightarrow \) (II) \( \Leftrightarrow \) (III)

We need the following lemma for the proof of the theorem.

LEMMA 2. Let \( ||A|| < \infty \), \( \phi \in \{1, R\} \), \( s: l_{\infty} \to l_{\infty} \) is the shift operator i.e.

\[
s(x_n) = x_{n+1}, \quad s^2(x_n) = x_{n+2}.
\]

Then

(a) \( |\phi(SX) - \phi(x)| \leq ||x|| \lim_{n \to \infty} \sup_{k \geq 0} \sup_{i} d_{n,k}^{(i)} \)

where \( d_{n,k}^{(i)} \) is defined by (1.8).

Let \( A \) be translative, then for \( x \in l_{\infty} \).

(b) (i) \( R(SX - x) = R(x - SX) = 0 \)

(ii) \( \phi(Sx) = \phi(x) \)

(c) \( R(Sx) = R(x) \)

Let, further

\[
\lim_{n \to \infty} a_{n,k}^{(i)} = 0, \text{ fixed } k, \text{ uniformly in } i. \quad (2.1)
\]

Then

(d) \( \lim_{n \to 0} s^{r_j} \phi_x = p.R(X) \)

Where \( r_0 = 1, r_1, r_2 \ldots \) \( r_p \) is a sequence of fixed positive integers.

PROOF: Since

\[
R(Sx-x) = \sup_{n \to \infty} \sup_{k \geq 0} a_{n,k}^{(i)} (Sx_k - x_k)
\]
It follows that
\[
|R(Sx-x)| \leq |x| \quad \limsup_{n \to \infty} \sup_{k \geq 0} d_{n,k}^{(1)} |x_k| .
\]

(2.2)

Now as \( \phi \) is linear, we obtain
\[
\phi(sx) - \phi(x) = \phi(sx-x) \leq R(sx-x).
\]

(2.3)

Changing the role of \( sx \) and \( x \) in (2.2) and (2.3) we obtain (a). When \( A \) is transitive (b) (i), (ii) follows from (2.2), and changing the role of \( sx \) and \( x \) in (2.2) (b) (ii) follows from (a). Since, \( R \) is sublinear,
\[
R(Sx) = R(Sx-x+x) \leq R(Sx-x) + R(x) = R(x)
\]

by (2.1). Changing the role of \( Sx \) and \( x \), we obtain \( R(x) \leq R(Sx) \). So (c) follows.

Lastly \( R(S^{r_1}x + S^{r_2}x) = R(S^{r_1}x - x + S^{r_2}x - x + 2x) \leq R(S^{r_1}x-x) + R(S^{r_2}x-x) + 2R(x) \).

(2.3)

i.e.
\[
R(S^{r_1}x + S^{r_2}x) = R(S^{r_1}x - x) + R(S^{r_2}x - x)
\]

(2.3)

But,
\[
R(S^{r_1}x - x) = \limsup_{n \to \infty} \sup_{k \geq 0} d_{n,k}^{(1)} (x_{k+r_1} - x_k)
\]

= \limsup_{n \to \infty} \sup_{k \geq r_1} a_{n,k}^{(i)} (x_{k+r_1} - x_k)

= \limsup_{n \to \infty} \sup_{k \geq r_1} [a_{n,k-r_1}^{(i)} - a_{n,k}^{(i)}] x_k

= \limsup_{n \to \infty} \sup_{k \geq r_1} [a_{n,k-r_1}^{(i)} - a_{n,k}^{(i)}] x_k

= \limsup_{n \to \infty} \sup_{j \geq r_1} [a_{n,k}^{(i)} - a_{n,k-j}^{(i)}] x_k

\leq |x| \quad \limsup_{n \to \infty} \sup_{k \geq r_1} d_{n,k}^{(1)}

= 0 \quad (\because A \text{ is transitive})

Similarly
\[
R(S^{r_2}x - x) \leq 0.
\]

Hence,
\[
R(S^{r_1}x + S^{r_2}x) \leq 2R(x), \quad x \in L_\infty .
\]

Again, since
\[
2R(x) = R(2x - S^{r_1}x - S^{r_2}x) \leq R(x - S^{r_1}x) + R(S^{r_1}x - S^{r_2}x) + R(S^{r_1}x + S^{r_2}x) .
\]

Proceeding as above, we have
\[
2R(x) \leq R(S^{r_1}x + S^{r_2}x), \quad x \in L_\infty .
\]

Hence,
\[
R(S^{r_1}x + S^{r_2}x) = 2R(x), \quad x \in L_\infty . \quad (2.4)
\]

(d) follows by repeated application of (2.4).
PROOF OF THEOREM I.

(a) Let (II) hold. Then

\[-R \left[ -m(T^{-n}B \cap C) \right] = R \left[ m(T^{-n}B \cap C) \right] \]

Since

\[-R(x) \leq Q(x) \leq R(x) , \quad x \in L_{\infty} \] (2.5)

It follows that

\[ \phi \left[ m(T^{-n}B \cap C) \right] = m(B) \cdot m(C) , \quad n = 0, 1, 2, \ldots \]

This proves (II) \(\Rightarrow\) (I).

(b) Take, \( T^{-1}B = B \), \( C = x/B = B^{-1} \) in (I).

Hence it follows that

\[ 0 = \phi(0) = m(B) \cdot m(B^{-1}) \]

either \( m(B) = 0 \) or \( m(B^{-1}) = 0 \).

i.e. \( T \) is ergodic.

Writing, \( C = X \) in (I), we obtain

\[ \phi \left[ m(T^{-n}B) \right] = m(B) \cdot m(X) \] (2.6)

Replacing \( B \) by \( T^{-1}B \) in (2.6), we obtain

\[ \phi \left[ m(T^{n-1}B) \right] = m(T^{-n}B) \cdot m(X) \] (2.7)

If further, \( A \) is translatable, by Lemma 2 (b)

\[ \phi \left[ m(T^{n-1}B) \right] = \phi \left[ m(T^{-n}B) \right] \]

Again, since \( 0 < m(X) < \infty \), it follows from (2.6) and (2.7) that

\[ m(T^{-1}B) = m(B) \]

Hence, (I) \(\Rightarrow\) (III).

(c) In view of (a) and (b), it is enough to show that (III) \(\Rightarrow\) (II). Take any fixed \( B \in F \) such that \( m(B) > 0 \). Define, for \( \phi \in \{ 1, \infty \} \) and \( C \in F \)

\[ q_n(c) = \frac{m(T^{-n}B \cap C)}{m(B)} , \quad n = 0, 1, 2, \ldots \]

\[ q(c) = \phi(q_n(c)) \] (2.8)

We now show that \( q \) is an invariant measure and \( m = q \).

Since, \( A \) is almost positive

\[ \lim_{n \to \infty} \frac{a_{n,k}^{-1}(x)}{k^{0}} x_k = 0 , \quad \text{uniformly in } i , \] (2.9)

for \( x_k \in L_{\infty} \).

Write

\[ R^{+}(x) = \lim_{n \to \infty} \sup_{k \geq 0} a_{n,k}^{+}(x) \cdot x_k . \]

So, by (2.9)

\[ R(x) = R^{+}(x) \]

Since, \( x \geq 0 \Rightarrow R^{+}(x) \geq 0 \)

\[ x \geq 0 \Rightarrow R(x) \geq 0 \] (2.10)

Again, since \( m \) is a measure, \( q_n(c) \geq 0 \). So it follows from (2.10) that

\[ R(q_n(c)) \geq 0 \]
Since $R$ is sublinear, we have

$$-R[-q_n(c)] \geq 0.$$  

Now, it follows from (2.5) that $q(c) \geq 0$, $c \in F$. Let $B_i \in F$ be a countable sequence of disjoint sets. Then

$$q(\bigcup_{i=1}^{\infty} B_i) = \phi \left[ \bigcup_{n=1}^{\infty} q_n(B_i) \right]$$

$$= \phi \left[ \bigoplus_{i=1}^{\infty} q_n(B_i) \right] \quad (\text{'.m is a measure})$$

$$= \bigoplus_{i=1}^{\infty} \phi \left[ q_n(B_i) \right] \quad (\phi \text{ is continuous linear functional})$$

So, $q$ is countably additive and hence it is a measure.

Next,

$$q(T^{-1}C) = \phi \left[ \frac{m(T^{-n}B\cap T^{-1}C)}{m(B)} \right]$$

$$= \phi \left[ \frac{m(T^{-n+1}B\cap C)}{m(B)} \right] \quad (\cdot T \text{ is a measure preserving})$$

$$= \phi \left[ q_{n-1}(C) \right]$$

Since $\phi$ is shift invariant by Lemma 2 (b),

$$= \phi \left[ q_n(C) \right]$$

$$= q(C), \quad C \in F.$$  

This proves that $q$ is an invariant measure. Since $T$ is ergodic, the invariant sets are of measure 0 or 1. Since $m$ and $q$ are invariant measures, an invariant measure is determined by the value it takes on invariant sets (See Sucheston [8], Theorem 2), it follows that $q = m$.

Now, we have

$$q(C) = m(C) = \phi \left[ \frac{m(T^{-n}B\cap C)}{m(B)} \right] \quad \text{i.e. } \phi \left[ m(T^{-n}B\cap C) \right] = m(B) \cdot m(C).$$

Hence, $\phi$ is unique on $\{T^{-n}B \cap C, n = 0,1,2, \ldots\}$. But, $\phi \in \{1, R\}$ has unique value $\lambda$ if and only if

$$R(x) = -R(-x) = \lambda.$$  

Hence, it follows that

$$R[m(T^{-n}B \cap C)] = -R[-m(T^{-n}B \cap C)] = m(B) \cdot m(C).$$

i.e. (II) holds and hence proves (c) completely.

3. EQUIVALENT MEASURES.

Many necessary and sufficient conditions have been determined for the existence of equivalent invariant measures (see Sucheston [7], [8], Mrs. Dowker [9], Calderon [10], and Hajian and Kakutani [11]). In the pointwise ergodic theorem of Birkhoff [12], it was necessary to take invariant measure, but Halmos [13] has shown that even if a measure is null invariant and conservative, an equivalent measure need not exist.

Sucheston [7], [8] has used Banach limit technique to prove the existence of invariant measures. We now generalize some of the theorems of Sucheston [5] involving almost convergence and some results of Mrs. Dowker on $(C, I)$ convergence and establish the existence of invariant measure by using linear functional $\phi \in \{1, R\}$.

We now prove

**THEOREM 2.** Let $A$ be a real matrix such that $||A|| < \infty$ and let $A$ be almost positive and translatable. Let $(x, F, m)$ be a finite measure space and $T$ be a measurable
transformation. Then, the following condition are equivalent.

(I) There exists an equivalent finite invariant measure.

(II) For some \( \phi \in \{1, \tau, R \} \) and all \( B \in F \)
\[
m(B) > 0 \implies \phi [m(T^{-n}B)] > 0
\]

(III) \( m(B) > 0 \implies R [m(T^{-n}B)] > 0 \)

PROOF. (I) \( \implies \) (II). Suppose that \( \rho \) is an invariant measure which is equivalent to \( m \). Suppose that (II) fails to hold. Then there exists a \( B \in F \) such that \( m(B) > 0 \) and
\[
\phi [m(T^{-n}B)] = 0.
\]
But, since \(-R(-x) \leq \phi(x) \leq R(x), \ x \in [1, \tau]\) and by Lemma 1
\[
\lim_{n \to \infty} x_n \leq -R(-x) \leq R(x) \leq \lim_{n \to \infty} x_n.
\]
it follows that for all \( B \in F \)
\[
0 = \phi [m(T^{-n}B)] \geq \lim_{n \to \infty} m(T^{-n}B).
\]
But, since \( \lim_{n \to \infty} m(T^{-n}B) \geq 0 \), it follows that
\[
\lim_{n \to \infty} m(T^{-n}B) = 0.
\]
Hence, there exists a sub sequence \( \{x_k\} \) such that
\[
\lim_{k \to \infty} m(T^{-nk}B) = 0.
\]
Since \( \rho \) is equivalent to \( m \), we obtain
\[
\rho(B) > 0 \text{ and } \lim_{k \to \infty} \rho(T^{-nk}B) = 0.
\]
Since \( \rho \) is invariant, we have
\[
\rho(T^{-nk}B) = \rho(B).
\]
Hence \( \rho(B) = 0 \). This is a contradiction and this proves the fact that (I) \( \implies \) (II).

(II) \( \implies \) (III). Let (II) hold. Since, \( \phi [m(T^{-n}B)] \leq R [m(T^{-n}B)] \) it follows that
\[
\phi [m(T^{-n}B)] > 0 \implies R [m(T^{-n}B)] > 0.
\]

(III) \( \implies \) (I). Suppose (III) holds and (I) fails. Since Condition (I) is equivalent to non-existence of weakly wandering set (See Sucheston [7], Theorem 6) it follows that there exists positive integers \( r_0 = 1, r_1, r_2, \ldots \) and a set \( B \in F \) with \( m(B) > 0 \) such that
\[
B, T^{-r_1}B, T^{-r_2}B, \ldots, T^{-r_k}B, \ldots
\]
are mutually disjoint. Since,
\[
\lim_{n \to \infty} \frac{a_{n,k}}{(1)} = 1, \text{ uniformly in } i, \text{ it follows that}
\]
\[
R[m(T^{-k}X)] = R[m(X)] = m(X) R(1) = m(X).
\]
Again
\[
m(X) = R [m(T^{-n}X)] \geq R [m(T^{-j}B)]
\]
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$$m(X) = \limsup_{n \to \infty} \sup_{k \geq 0} \sum_{i=0}^{s} a_{n,k} m(T^{-j} B_k)$$

$$= \limsup_{n \to \infty} \sup_{k \geq 0} a_{n,k} S \sum_{j=0}^{s} r_j x_k$$

Where $X_k = m(T^{-k} B)$, $S$ is a shift operator. Then by Lemma 2(d),

$$m(X) \geq S \cdot R(X) \quad (3.1)$$

Then it follows from (3.1) that

$$m(X) \geq s \cdot R \cdot m(T^{-k} B) \quad (3.2)$$

Since $m(T^{-k} S) > 0$ by hypothesis and $s$ is an arbitrary positive integer. This contradicts (3.2). This proves (III) $\Rightarrow$ (I).

In the next theorem, we give yet another characterization of existence of invariant measure in terms of the sublinear functional $R(x)$.

**Theorem 3.** Let $A$ satisfy the condition of Theorem 2. Let $(X, F, m)$ be the finite measure space. Then there exists an invariant measure equivalent to measure $m$ on $X$, if and only if,

(i) $m$ is null-preserving.

(ii) $T$ is conservative.

(iii) $n \to 0 \sum_{i=0}^{s} a_{n,k} m(T^{-k} B)$ converges uniformly in $i$, for every $B \in F$.

Again, whenever it has equivalent invariant measure, then the map $q : F \to R$ defined by $q(B) = R \left[ m(T^{-n} B) \right]$ is an invariant measure equivalent to $m$ and agrees with $m$ on invariant sets.

**Proof:** Necessity.

Let us assume that $m$ admits an invariant equivalent measure $\mu$. Then $\mu$ is $m$ continuous (See Halmos [9] p. 125).

Write for $\phi \in \{1, 2\}$

$$q(B) = \phi \left[ m(T^{-n} B) \right]$$

We want to show

(a) $q$ is a measure

(b) $q$ is a $m$ continuous

(c) $q$ is invariant.

As in the proof of Theorem 1, we can show that

$$q(B) \geq 0, \text{ for all } B \in F.$$  

It is easy to show that

$$B, C \in F, \quad B \subseteq C \Rightarrow q(B) \leq q(C).$$

Since $\phi$ is linear, it also follows that $q$ is finitely additive. Since $\mu$ is $m$-continuous, for given $\epsilon > 0, \quad \delta > 0$. Such that

$$m(T^{-n} B) < \epsilon \quad \text{when} \quad \mu(B) = \mu(T^{-n} B) < \delta \quad \text{and} \quad m(T^{-n} B) < \epsilon \Rightarrow q(B) < \epsilon.$$  

So $q$ is $m$-continuous. The countably additivity of $m$ and $m$-continuity of $q$ (See Halmos [9] p. 39).

Next,

$$q(T^{-1} B) - q(B) = \phi \left[ m(T^{-n-1} B) \right] - \phi \left[ m(T^{-n} B) \right]$$

$$= \phi \left[ m(T^{-n-1} B) - m(T^{-n} B) \right] \quad (\phi \text{ is linear}).$$
\[ \leq R [m(T^{-n-1}B) - m(T^{-n}B)] = \lim_{n \to \infty} \sup_{k \geq 0} a_{n,k} \left[ (i) \right] m(T^{-k}B) \]

Since \( a_{n-1} (1) = 0 \) for all \( n \) and \( i \)

\[ = \lim_{n \to \infty} \sup_{k \geq 0} \left[ a_{n,k-1} - a_{n,k} \right] m(T^{-k}B) \]

\[ \leq m(X) \lim_{n \to \infty} \sup_{k \geq 0} \left| d_{n,k} \right| . \]

Since \( A \) is transitive,

\[ \implies 0 \text{ as } n \to \infty, \text{ uniformly in } i. \]

Hence,

\[ q(T^{-1}B) \leq q(B). \]

Changing the role of \( T^{-1}B \) and \( B \), we obtain

\[ q(B) \leq q(T^{-1}B). \]

Hence,

\[ q(T^{-1}B) = q(B) \]

i.e. \( q \) is invariant under \( T \).

Now if \( T^{-1}B = B \). Then,

\[ q(B) = \phi(m(T^{-1}B)) = \phi(m(B)), \]

\[ = m(B) \phi(1) = m(B). \]

So \( q = m \) on invariant sets. Hence (Sucheston [8], Theorem 2) \( q = m \) on \( F \). Thus \( q(B) = \phi(m(T^{-n}B)) \) is unique. But \( \phi \in \{1, R\} \) is unique if and only if \( R(x) = -R(-x) \) = \( q(B) \) and this happens if and only if

\[ \lim_{n \to \infty} \sup_{k \geq 0} a_{n,k} \left[ (i) \right] m(T^{-k}B) = q(B) \]

uniformly in \( i \).

Now since, \( q(T^{-1}B) = q(B) \), \( B \in F \) and \( q = m \) on \( F \), we have \( m(T^{-1}B) = m(B) \), \( B \in F \) so \( m(B) = 0 \implies m(T^{-1}B) = 0 \)

i.e. \( m \) is null-preserving.

Again (See Sucheston [7], Theorem 6) existence of invariant measure is equivalent to non-existence of weakly wandering sets and non-existence of weakly wandering sets is the same as conservativeness of \( T \).

**SUFFICIENCY:**

Let (i), (ii) and (iii) hold. Define

\[ q(B) = \lim_{n \to \infty} \sup_{k \geq 0} a_{n,k} \left[ (i) \right] m(T^{-k}B) \]

Then it can be proved as before that \( q \) is an invariant measure. So only we have to prove \( q \) is equivalent to \( m \). Since \( T \) is null preserving,

\[ m(B) = 0 \implies m(T^{-1}B) = 0. \]

Then

\[ q(B) = \lim_{n \to \infty} \sup_{k \geq 0} a_{n,k} \left[ (i) \right] m(T^{-k}B) = 0 \]

uniformly in \( i \).
Conversely, let \( q(B) = 0 \).

Write:

\[
A^* = \bigcup_{n=1}^{\infty} \bigcup_{i=n}^{\infty} T^{-i}B.
\]

Then

\[
q(A^*) = \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} q(T^{-i}B)
\]

\[
\leq \sum_{i=1}^{\infty} \sum_{n=1}^{i} q(T^{-i}B) (q \text{ is a measure})
\]

\[
= \sum_{i=1}^{\infty} q(B) (q \text{ is invariant})
\]

\[
= 0.
\]

Since, \( q \) and \( m \) agrees on invariant sets, we have \( m(A^*) = 0 \). Since, \( T \) is conservative by recurrence theorem \( m(B/A^*) = 0 \implies m(B) = 0 \).

Hence \( q \) is equivalent to \( m \).

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