ABSTRACT. Let \( F(p, \beta, M) \left( 0 < p < 1, |a| < \frac{\pi}{2}, 0 < \beta < 1 \text{ and } M > \frac{1}{2} \right) \), denote the class of functions \( f(z) \) which are regular in \( U = \{z:|z| < 1\} \) and of the form
\[
f(z) = z + a_2 e^{\frac{-2a_2 z^2}{1-\beta}} e^{\frac{ia}{1-\beta} \cos \alpha} + \ldots,
\]
where \( |a_2| = p(1+\sigma)(1-\beta)\cos \alpha \), which satisfy for fixed \( M \), \( z \in U \) and
\[
\left| \frac{\frac{i\alpha}{1-\beta}zf'(z)}{f(z) - \beta \cos \alpha - i \sin \alpha} - M \right| < M.
\]

In this paper we have found the sharp radius of \( \gamma \)-spiralleness of the functions belonging to the class \( F(p, \beta, M) \).

KEY WORDS AND PHRASES. Spiral-like, bounded functions, radius of \( \gamma \)-spiralleness.

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1. INTRODUCTION. Let \( A \) denote the class of functions which are regular and univalent in the unit disc \( U = \{z:|z| < 1\} \) and satisfy the conditions \( f(0) = 0 = f'(0)-1 \).

Let \( F(\alpha, \beta, M)(|a| < \frac{\pi}{2}, 0 < \beta < 1 \text{ and } M > \frac{1}{2}) \) denote the class of bounded \( \alpha \)-spiral-like functions of order \( \beta \), that is \( f \in F(\alpha, \beta, M) \) if and only if for fixed \( M \),
\[
\left| \frac{\frac{i\alpha}{1-\beta}zf'(z)}{f(z) - \beta \cos \alpha - i \sin \alpha} - M \right| < M, \quad z \in U.
\]

The class \( F(\alpha, \beta, M) \) introduced by Aouf [1], he proved that if \( f(z) = z + a_2 z^2 + \ldots \in F(\alpha, \beta, M) \) then,
\[
|a_2| < (1+\sigma)(1-\beta)\cos \alpha, \quad \sigma = 1 - \frac{1}{M}.
\]
If $c = \exp(-i \arg a_2 - i\alpha)$, then 
\[
\frac{f(\epsilon z)}{\epsilon} = z + a_2 e^{-ia_2 z^2} + \ldots \in F(\alpha, \beta, M),
\]
whenever $f(z) \in F(\alpha, \beta, M)$. Thus without loss of generality we can replace the second coefficient $a_2$ of $f(z) \in F(\alpha, \beta, M)$ by $a_2 e^{-ia_2}$.

Let $F_p(\alpha, \beta, M)$ denote the class of functions $f(z) = z + a_2 e^{-ia_2 z^2} + \ldots$, which satisfy (1.1), where $|a_2| = p(1 + o)(1 - \beta) \cos \alpha$. In view of (1.2) it follows that $0 < p < 1$.

Let $G_p(\alpha, \beta, M)$ denote the class of functions $g(z) = z + b_2 e^{-ia_2 z^2} + \ldots$, regular in $U$ and satisfy the condition
\[
\left| \frac{e^{-ia}(1 + \frac{g''(z)}{g'(z)}) - \beta \cos a \sin \alpha}{(1-\beta) \cos \alpha} \right| < M, z \in U,
\]
where $|b_2| = \frac{1}{2} p(1 + o)(1 - \beta) \cos \alpha$.

It follows from (1.1) and (1.3) that
\[
g(z) \in G_p(\alpha, \beta, M), \text{ if and only if } zg'(z) \in F_p(\alpha, \beta, M).
\]

We note that by giving specific values to $p, \alpha, \beta, M$, we obtain the following important subclasses studied by various authors in earlier papers:

(i) $F_1(\alpha, \beta, M) = F_M(\alpha, \beta)$ and $G_1(\alpha, \beta, M) = G_M(\alpha, \beta)$, are respectively the class of bounded spiral-like functions of order $\beta$ and the class of bounded Robertson functions of order $\beta$ investigated by Aouf [1] and $F_1(\alpha, 0, M) = F_{\alpha_M}$ and $G_1(\alpha, 0, M) = G_{\alpha_M}$, are respectively the class of bounded spiral-like functions and the class of bounded Robertson functions investigated by Kulshreshtha [2].

(ii) $F_p(\alpha, \beta, \sigma) = F_p(\alpha, \beta)$ and $G_p(\alpha, \beta, \sigma) = G_p(\alpha, \beta)$, are considered by Umarani [3].

In this paper we determine the sharp radius of $\gamma$-spiralness of the functions belonging to the class $F_p(\alpha, \beta, M)$, generalizing an earlier result due to Kulshreshtha [2], Libera [4], Umarani [5, 3].

The technique employed to obtain this result is similar to that used by McCarty [6] and Umarani [3].

2. THE SHARP RADIUS OF $\gamma$-SPIRALNESS OF THE CLASS $F_p(\alpha, \beta, M)$, $M > 1$.

**Lemma 1.** If $f(z) \in F_p(\alpha, \beta, M) M > 1$, then
\[
|\frac{zf''(z)}{f'(z)} - \omega| < \rho_0,
\]
where
\[
\rho_0 = \frac{1}{p} \left( \frac{1}{1 - \beta} \right) \left( \frac{1}{1 + o} \right) \cos \alpha,
\]
and $f(z)$ is analytic in $U$. If $f(z)$ satisfies (1.1), then the sharp radius of $\gamma$-spiralness of $f(z)$ is $\rho_0$.
where

\[ w_o = \frac{(1+pr)^2 + [(1-\beta)(\frac{1+\sigma}{\sigma}) - 1] \cos \alpha - i \sin \alpha} {1 - r^2 + 2pr + r^2} \]  

(2.2)

and

\[ \rho_o = \frac{(1+\sigma)(1-\beta)\cos \alpha}{(1-r^2)(1+2pr + r^2)} \]  

(2.3)

This result is sharp.

**Proof.** Let \( f(z) \in F(\alpha, \beta, M), M > 1 \). Then there exists a function \( w(z) \) analytic in \( U \) and \( |w(z)| < 1 \) in \( U \) such that

\[ e^{\frac{i\alpha}{f(z)}} \frac{zf'(z)}{f(z)} = \cos \alpha \left( \frac{1 + [(1-\beta)(\frac{1+\sigma}{\sigma}) - 1] \omega(z)} {1 - \sigma \omega(z)} \right) + i \sin \alpha, \sigma = 1 - \frac{1}{M} \]

or

\[ \frac{zf'(z)}{f(z)} = \frac{1 + [(1-\beta)(\frac{1+\sigma}{\sigma}) - 1] \cos \alpha - i \sin \alpha} {1 - \sigma \omega(z)} \]  

Solving for \( w(z) \),

\[ w(z) = \frac{zf'(z)}{f(z)} - \frac{1}{\sigma \left[ zf'(z) + [(1-\beta)(\frac{1+\sigma}{\sigma}) - 1] \cos \alpha - i \sin \alpha \right] e^{-i\alpha} \sigma} \]

Since \( f(z) = z + 2^2 e^{-i\alpha z^2} + \ldots \), we obtain \( w(z) = p + \ldots = z \phi(z) \), where \( \phi(z) \) is analytic in \( U \), \( \phi(0) = p \) and \( |\phi(z)| < 1 \) in \( U \). Now \( \frac{\phi(z) - p}{1 - p \phi(z)} = z \). Therefore

\[ \phi(z) = \frac{z + p}{1 + pz} \]

Also \( |w(z)| = |z\phi(z)| < \frac{|z| + p}{1 + |z| p} |z| \). Let \( g(z) = \frac{|z| + p}{1 + p|z|} \) and

\[ h(z) = \frac{1 + [(1-\beta)(\frac{1+\sigma}{\sigma}) - 1] \cos \alpha - i \sin \alpha} {1 - \sigma z} \]

Since the image of \( |z| < r \) under \( g(z) \) is a disc and \( h(z) \) is a bilinear transformation, then \( \frac{zf'(z)}{f(z)} \) is subordinate to \( (hog)(z) \). That is, the image of \( |z| < r \) under \( \frac{zf'(z)}{f(z)} \) is contained in the image of \( |z| < r \) under \( (hog)(z) \).

Equality in (2.1) can be attained by a function

\[ f(z) = z(1-2p\sigma + \sigma^2) \]

(2.4)
\[ f(z) = z + p(1+\sigma)(1-\beta)\cos \alpha e^{-i\sigma z^2} + \ldots \]

hence

\[
\frac{zf''(z)}{f'(z)} = \frac{1 - 2p\sigma z + \sigma z^2 - (1+\sigma)(1-\beta)\cos \alpha e^{-i\sigma z^2}}{1 - 2p \sigma z + \sigma z^2} = \frac{1 + \sigma \psi(1+\sigma)(1-\beta)\cos \alpha e^{-i\sigma \psi}}{1 + \sigma \psi},
\]

where

\[ \psi = \frac{\pi(z-p)}{1-\rho_0 z}. \]

Since \( p < 1, \sigma < 1, |\psi| < 1 \) for \( z \in U \).

This shows that

\[
e^{-i\sigma \psi} \frac{zf''(z)}{f'(z)} = \cos \alpha \left( \frac{1 + (1+\sigma)(1-\beta)\psi(z)}{1 + \sigma \psi(z)} \right) + i \sin \alpha
\]

and

\[
e^{-i\sigma \psi} \frac{zf''(z)}{f'(z)} - i \sin \beta \cos \alpha \frac{1 - \psi(z)}{(1-\beta)\cos \alpha} = \frac{1 - \psi(z)}{1 + \sigma \psi(z)}.
\]

Then it is easy to show that \( \left| \frac{1 - \psi(z)}{1 + \sigma \psi(z)} - M \right| < M, \sigma = 1 - \frac{1}{M} \). Thus \( f \in F_p(\alpha,\beta,M) \).

Substituting \( \psi = \frac{\delta \sigma e^{-i\sigma \psi}}{\sigma (1 - \sigma e^{-i\sigma \psi})} \), where \( \delta = \frac{r(r+p)}{1+rp} \) in (2.5), we find that

\[
|zf''(z)| = \omega_0 \rho_o, \text{ where } \omega_0 \text{ and } \rho_o \text{ are given by (2.2) and (2.3).}
\]

This completes the proof of the lemma.

**Remark 1.**

(i) If \( p=1 \) and \( \beta=0 \) in Lemma 1, we obtain a result of Kulshrestha [2].

(ii) If \( M=\infty(\sigma=1) \) in Lemma 1, we obtain a result of Umarani [3].

(iii) If \( \alpha=0 \) and \( M=\infty(\sigma=1) \) in Lemma 1, we obtain a result of McCarty [6].

**Theorem 1.** If \( f(z) \in F_p(\alpha,\beta,M), \) then \( f(z) \) is \( \gamma \)-spiral if

\[
|z| < r_\gamma, \text{ where } r_\gamma \text{ is the smallest positive root of the equation}
\]

\[
\cos \gamma + p \left[ 2 \cos \gamma - (1+\sigma)(1-\beta)\cos \alpha r + [p^2 \cos \gamma + \sigma^2 \cos \gamma - (1+\sigma)(1-\beta)\cos \alpha (1+p^2)] r^2 + p [2c-(1+\sigma)(1-\beta)\cos \alpha r^3 + cr^4 = 0, \right]
\]

(2.6)
where \( c = \cos(\gamma-2a) + [(1-\beta)(\frac{1+\sigma}{\sigma})-2] \cos \alpha \cos(\gamma-a) \). The result is sharp.

**Proof.** Let \( f(z) \in \mathcal{F}_p(\alpha,\beta,M), M > 1 \), then by the above Lemma, we have

\[
\left| \frac{zf'(z)}{f(z)} - w_0 \right| < \rho_0.
\]

Hence \( \Re e^{i\gamma} \frac{zf'(z)}{f(z)} > \Re e^{i\gamma} \cdot w_0 - \rho_0 \)

\[
\cos \gamma (1+pr)^2 + \Re \left[ \left(1-\beta(\frac{1+\sigma}{\sigma})-1\right) \cos a \sin a \right] e^{i(\gamma-a)} r^2 (r+p)^2
\]

\[
= \frac{\left[ -(1+\sigma)(1-\beta) \cos a (1+pr)(r+p) \right]}{(1-r^2)(1+2pr+r^2)}.
\]

Thus, \( f(z) \) is \( \gamma \)-spiral if the R.H.S. of (2.7) is positive. Hence \( f(z) \) is \( \gamma \)-spiral for \( |z| < r_\gamma \), where \( r_\gamma \) is the smallest positive root of the equation

\[
\cos \gamma (1+pr)^2 + \left[ \cos(\gamma-2a) + [(1-\beta)(\frac{1+\sigma}{\sigma})-2] \cos \alpha \cos(\gamma-a) \right] r^2 (r+p)^2
\]

\[
-\frac{-(1+\sigma)(1-\beta) \cos a (1+pr)(r+p)}{(1-r^2)(1+2pr+r^2)} = 0.
\]

Simplifying the above equation, we obtain (2.6).

If \( \gamma = 0 \) in the above Theorem, we obtain the radius of starlikeness of the class \( \mathcal{F}_p(\alpha,\beta,M) \).

**Corollary 1.** \( f(z) \in \mathcal{F}_p(\alpha,\beta,M), M > 1 \), is starlike for \( |z| < r_0 \), where \( r_0 \) is the least positive root of the equation

\[
1+p [2-(1+\sigma)(1-\beta) \cos a] r +
\]

\[
(\frac{1+\sigma}{\sigma})(1-\beta) \cos a \left[ \cos \alpha (1+p^2) - \sigma (1+p^2) \right] r^2 +
\]

\[
p[2c-(1+\sigma)(1-\beta) \cos a] r^3 + cr^4 = 0,
\]

where \( c = (\frac{1+\sigma}{\sigma})(1-\beta) \cos^2 a - 1 \).

If \( p=1, \gamma=0 \) and \( \beta=0 \) in Theorem 1, we obtain a result of Kulshrestha [2].

**Corollary 2.** \( f(z) \in \mathcal{F}_p(\alpha,\beta,1), M > 1 \), is starlike for \( |z| < r_0 \), where \( r_0 \) is the least positive root of the equation

\[
1-(1+\sigma) \cos a r + \left[ (\frac{1+\sigma}{\sigma}) \cos^2 a - 1 \right] r^2 = 0.
\]
REMARK 2.

(i) If $M = (\sigma=1)$ in Theorem 1, we obtain a result of Umarani [3].

(ii) If $p=1$ and $M = (\sigma=1)$ in Theorem 1, we obtain a result of Libera [4] and Umarani [5].

(iii) If $p=1$, $\beta=0$, $\gamma=0$ and $M = (\sigma=1)$ in Theorem 1, we obtain a result of Robertson [7].

Since $g(z) \in G_\omega^p(\alpha,\beta,M)$ if and only if $z\frac{g'(z)}{g(z)} \in \mathcal{F}_\omega^p(\alpha,\beta,M)$ we obtain from Theorem 1,

THEOREM 2. If $g(z) \in G_\omega^p(\alpha,\beta,M)$, $M > 1$, then $\Re e^{\frac{1}{\gamma}(1 + \frac{zg''(z)}{g'(z)})} > 0$ for $|z| < r_\gamma$, where $r_\gamma$ is the least positive root of equation (2.6).

The result is sharp.

If $\gamma=0$ in Theorem 2, we obtain the radius of convexity of the class $G_\omega^p(\alpha,\beta,M)$.

COROLLARY 3. If $g(z) \in G_\omega^p(\alpha,\beta,M)$, $M > 1$, then the radius of convexity of $g(z)$ is the least positive root of equation (2.8).

REMARK 3.

(i) For $M = (\sigma=1)$ in Theorem 2, and Corollary 3, we obtain a result of Umarani [3].

(ii) If $p=1$ and $\beta=0$ in Corollary 3, we obtain a result of Kulshrestha [2].

(iii) For $p=1$ and $M = (\sigma=1)$, Theorem 2, generalizes the result of Umarani [5].

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