LOCALY CLOSED SETS AND LC-CONTINUOUS FUNCTIONS

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(Received March 24, 1988)

ABSTRACT. In this paper we introduce and study three different notions of generalized continuity, namely LC-irresoluteness, LC-continuity and sub-LC-continuity. All three notions are defined by using the concept of a locally closed set. A subset $S$ of a topological space $X$ is locally closed if it is the intersection of an open and a closed set. We discuss some properties of these functions and show that a function between topological spaces is continuous if and only if it is sub-LC-continuous and nearly continuous in the sense of Ptak. Several examples are provided to illustrate the behavior of these new classes of functions.

KEY WORDS AND PHRASES. Locally closed set, LC-continuous, nearly continuous.

1980 AMS SUBJECT CLASSIFICATION CODES. 54C10, 54A05.

1. INTRODUCTION.

In 1921 Kuratowski and Sierpinski [1] considered the difference of two closed subsets of an $n$-dimensional euclidean space. Implicit in their work is the notion of a locally closed subset of a topological space $(X,T)$. Following Bourbaki [2], we say that a subset of $(X,T)$ is locally closed in $X$ if it is the intersection of an open subset of $X$ and a closed subset of $X$. Stone [3] has used the term FG for a locally closed subset.

The following results indicate that locally closed subsets are of some interest in the setting of local compactness, Cech-Stone compactifications, or Cech complete spaces. From Engelking [4] we have:

1. If $(X,T)$ is Hausdorff and $C$ is a locally compact subspace of $X$, then $C$ is locally closed [4, p. 140, Ex. A].
2. If $(X,T)$ is locally compact and Hausdorff and $C \subseteq X$, then $C$ is locally compact if and only if $C$ is locally closed [4, p. 140, Ex. B].
3. If $X$ is completely regular, then $X$ is locally closed in $BX$ if and only if $X$ is locally compact [4, follows from Theorem 6, page 137].
4. If $X$ is completely regular and Čech complete and $C$ is locally closed in $X$, then 
$C$ is Čech complete [4, follows from Theorem 3, page 144].

Stone [3] has studied the absolutely FG spaces—the spaces that in every 
embedding are locally closed. He has shown that, for Hausdorff spaces, the 
hereditarily absolute FG spaces coincide with the hereditarily locally compact spaces.

The results of Borges [5] show that locally closed sets play an important role in 
the context of simple extensions. For example, if $T(A)$ denotes the simple extension 
of $T$ by $A$, then if $(X,T)$ is regular and $A \subseteq X$, we have that $(X, T(A))$ is regular if 
and only if $A$ is locally closed in $X$ [5, Theorem 3.2].

In 1922 Blumberg [6] introduced the concept of a real valued function on 
euclidean space being densely approached at a point of its domain. This notion was 
generalized in 1958 to general topological spaces by Ptak [7] who used the term nearly 
continuous. The concepts of nearly continuous and nearly open functions are important 
in functional analysis especially in the context of open mapping and closed graph 
theorems. We refer the reader to the work of Ptak [7], Pettis [8], Noll [9] and 
Wilhelm [10], [11], for example.

In this paper we consider a new class of generalized continuous functions which 
are called LC-continuous functions. Such a function is defined by requiring the 
inverse image of each open set in the codomain to be locally closed in the domain. 
The significance of this notion is that LC-continuity is the continuity dual of nearly 
continuity, that is a function is continuous if and only if it is nearly continuous 
and LC-continuous. This theorem enables us to obtain interesting variations of 
results from functional analysis. We quote two to illustrate. If $G$ is a Baire 
topological group and $H$ is a separable (or Hausdorff and Lindelof) topological group, 
then a homomorphism $f : G \rightarrow H$ is continuous if and only if it is sub-LC-continuous, 
Husain [13, p. 222]. If $X$ is a Baire topological vector space, $Y$ is a topological 
vector space and $f : X \rightarrow Y$ is linear, then $f$ is continuous if and only if it is sub-LC- 
continuous, Husain [13, p. 224].

In section 2 we consider the properties of locally closed subsets, while section 
3 introduces the classes of LC-irresolute, LC-continuous and sub-LC-continuous 
functions. Section 4 is concerned with some of the properties of these functions, and 
relevant examples are provided in section 5. We note that throughout this paper no 
separation properties are assumed unless explicitly stated.

2. LOCALLY CLOSED SETS.

Let $S$ be a subset of a topological space $(X,T)$. We denote the closure of $S$, the 
interior of $S$, and the boundary of $S$ with respect to $T$ by $T \overline{S}$, $T \text{int} S$, and $T \text{bd} S$ 
respectively, usually suppressing the $T$ when there is no possibility of confusion. 
The relative topology on $S$ with respect to $T$ is denoted by $T/S$. We will denote the 
set of all reals by $\mathbb{R}$ and the set of all positive integers by $\mathbb{N}$. Unless otherwise 
mentioned $\mathbb{R}$ carries its usual topology.

DEFINITION [2]. A subset $S$ of a space $(X,T)$ is called locally closed if 
$S = U \cup F$ 
where $U \in T$ and $F$ is closed in $(X,T)$. 

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We denote the collection of all locally closed subsets of \((X,T)\) by \(LC(X,T)\).

**REMARKS.**

(i) A subset \(S\) of \((X,T)\) is locally closed if and only if \(X-S\) is the union of an open set and a closed set.

(ii) Every open [resp. closed] subset of \((X,T)\) is locally closed. (iii) For any space \((X,T)\), \(LC(X,T)\) is closed under finite intersections. In particular, any interval in \(\mathbb{R}\) is locally closed. (iv) The complement of a locally closed subset need not be locally closed. Hence the finite union of locally closed subsets need not be locally closed (see e.g. Corollary 2). (v) A subset \(S\) of a space \((X,T)\) is said to be nearly open if \(S \subseteq \text{int}(cl\ S)\). Nearly open sets are known also as preopen sets [14]. Ganster and Reilly [12] have shown that a subset \(S\) of \((X,T)\) is open if and only if it is nearly open and locally closed. In particular, a dense subset is open if and only if it is locally closed. (vi) Spaces in which every singleton is locally closed are called \(T_D\) spaces [15].

The following result is essentially a restatement of I.3.3, Proposition 5, of [2].

**PROPOSITION 1** [2]. For a subset \(S\) of a space \((X,T)\) the following are equivalent.

(i) \(S\) is locally closed.

(ii) \(S = U \cap cl\ S\) for some open set \(U\).

(iii) \(cl\ S - S\) is closed.

(iv) \(S \cup (X - cl\ S)\) is open.

(v) \(S \subseteq \text{int}(S \cup (X - cl\ S))\).

Recall that \((X,T)\) is called submaximal if every dense subset is open. Using (iv) of Proposition 1 we immediately get

**COROLLARY 1.** A space \((X,T)\) is submaximal if and only if every subset of \((X,T)\) is locally closed.

The next result indicates where to look in order to find locally closed sets besides open sets and closed sets.

**PROPOSITION 2.**

(i) Let \((X,T)\) be a \(T_1\) space and let \(S\) be a discrete subset of \((X,T)\). Then \(S\) is locally closed. (ii) Let \((X,T)\) be dense-in-itself and \(S\) be a discrete subset. Then \(X-S\) is locally closed if and only if \(S\) is closed.

**PROOF.** Let \(S\) be a discrete subset of the \(T_1\) space \((X,T)\), i.e. for each \(x \in S\) there is an open set \(U_x\) such that \(U_x \cap S = \{x\}\). If \(U = \bigcup\{U_x | x \in S\}\) then it is easily verified that \(S = U \cap cl\ S\). This proves (i). In order to prove (ii), observe that in a dense-in-itself space any discrete subset has empty interior.

**COROLLARY 2.** If \(S = \{1/n | n \in \mathbb{N}\}\) then \(S\) is locally closed in \(\mathbb{R}\) whereas \(\mathbb{R} - S\) is not.

Note that in Proposition 2 (ii) the assumption that \((X,T)\) is dense-in-itself cannot be dropped. Consider a space \((X,T)\) whose set \(D\) of isolated points is a proper dense subset. Then clearly \(D\) is a nonclosed discrete subset whereas \(X - D\) is closed hence locally closed.

Our next four results exhibit some of the basic properties of locally closed sets.
PROPOSITION 3. Let \((X, T)\) be a space and let \(Z \in \text{LC}(X, T)\). If \(A \in Z\) and \(A \in \text{LC}(Z, T/Z)\) then \(A \in \text{LC}(X, T)\).

PROPOSITION 4. Let \(A\) and \(B\) be locally closed subsets of a space \((X, T)\). If \(A\) and \(B\) are separated, i.e. if \(A \cap \text{cl} B = \text{cl} A \cap \text{cl} B = \emptyset\), then \(A \cup B \in \text{LC}(X, T)\).

**Proof.** Suppose there are open sets \(U\) and \(V\) such that \(A = U \cap \text{cl} A\) and \(B = V \cap \text{cl} B\). Since \(A\) and \(B\) are separated we may assume that \(U \cap \text{cl} B = V \cap \text{cl} A = \emptyset\). Consequently \(A \cup B = (U \cup V) \cap \text{cl} (A \cup B)\) showing that \(A \cup B \in \text{LC}(X, T)\).

**Theorem 1.** Let \(\{Z_i \mid i \in I\}\) be either an open cover or a locally finite closed cover of a space \((X, T)\) and let \(A \subseteq X\). If \(A \cap Z_i \in \text{LC}(Z_i, T/Z_i)\) for each \(i \in I\) then \(A \in \text{LC}(X, T)\).

**Proof.** First suppose that \(\{Z_i \mid i \in I\}\) is an open cover of \((X, T)\). For each \(i \in I\), since \(A \cap Z_i \in \text{LC}(Z_i, T/Z_i)\) we may assume that \(A \cap Z_i = V_i \cap \text{cl}(A \cap Z_i)\) where \(V_i \subseteq T\) and \(V_i \subseteq Z_i\). Now \(V_i \cap \text{cl} A = V_i \cap Z_i \cap \text{cl} A = V_i \cap \text{cl}(A \cap Z_i) = A \cap Z_i\). Hence if \(V = U \cup \{V_i \mid i \in I\}\) we have \(V \cap \text{cl} A = A\).

Now suppose that \(\{Z_i \mid i \in I\}\) is a locally finite closed cover of \((X, T)\). For each \(i \in I\), since \(A \cap Z_i \in \text{LC}(Z_i, T/Z_i)\) we have \(A \cap Z_i = V_i \cap \text{cl}(A \cap Z_i)\) where \(V_i \subseteq T\). Let \(x \in A\). Since \(\{Z_i \mid i \in I\}\) is a locally finite closed cover, hence a point-finite and closure-preserving cover, there is a finite subset \(I_x \subseteq I\) such that \(x \in Z_i\) if \(i \in I_x\) and \(x \notin U_{Z_i} \cap I_{i - I_x}\). Moreover, there is an open set \(U_{Z_i}\) containing \(x\) such that \(U_{Z_i} \subseteq \cap \{V_i \mid i \in I_x\}\) and \(U_{Z_i} \cap \{\{Z_i \mid i \in I_{i - I_x}\} = \emptyset\) if \(U_{Z_i} \cap \text{cl} A = \emptyset\). Let \(y \in \text{cl} A\). Then \(y \in U_{Z_i}\) for some \(x \in A\). Since \(y \in \text{cl} A = U \cap \text{cl}(A \cap Z_j)\) for some \(j \in I\). Hence \(j \in I_x\) and \(j \in V_j\). Thus \(y \in V_j \cap \text{cl}(A \cap Z_j) = A \cap Z_j \subseteq A\). It follows that \(A = U \cap \text{cl} A\).

PROPOSITION 5. For each \(i \in I\), let \((X_i, T_i)\) be a space and let \(S_i \in \text{LC}(X_i, T_i)\). If \(S_i \subseteq X_i\) except for finitely many \(i \in I\), then \(S_i\) is a locally closed subset of the product space \(\prod X_i\).

In general one cannot expect that the set-theoretic complement of a locally closed set is locally closed. The following result characterizes those spaces in which a locally closed subset has necessarily a locally closed complement. Recall that a subset \(S\) of a space \((X, T)\) is said to be semi-open if \(S \subseteq \text{cl} (\text{int} S)\).

THEOREM 2. For a \(T_1\) space \((X, T)\) the following are equivalent:

(i) \(S \in \text{LC}(X, T)\) if and only if \(X-S \in \text{LC}(X, T)\).

(ii) \(\text{LC}(X, T)\) is closed under finite unions.

(iii) The boundary of each open set is a discrete subset.

(iv) The boundary of each semi-open set is a discrete subset.

(v) Every semi-open set is locally closed.

**Proof.** (i)\(\Rightarrow\)(ii) is obvious.

(ii)\(\Rightarrow\)(iii): Let \(U\) be open and let \(x \in \text{bd} U = \text{cl} U \cap (X-U)\). By assumption, if \(S = U \cup \{x\}\) then \(S \in \text{LC}(X, T)\). Let \(S = V \cap \text{cl} S\) for some open set \(V\). One easily verifies that \(V \cap \text{bd} U = \{x\}\).

(iii)\(\Rightarrow\)(iv): Let \(S\) be semi-open in \((X, T)\) and let \(U = \text{int} S\). Then \(\text{bd} S \subseteq \text{bd} U\) and hence \(\text{bd} S\) is a discrete subset.

(iv)\(\Rightarrow\)(v): Let \(S\) be semi-open in \((X, T)\). For each \(x \in S \cap \text{bd} S\) there is an open set \(U_x\) such that \(U_x \cap \text{bd} S = \{x\}\). If \(U = \text{int} S \cup \{U_x \mid x \in S \cap \text{bd} S\}\) it is easily verified that \(S = U \cap \text{cl} S\).
(v) \Rightarrow (i): We will show that any union of an open set and a closed set is locally closed. Let \( A = U \cup F \) where \( U \) is open and \( F \) is closed. We may assume that \( U \cap F = \phi \). If \( S = U \cup (cl U \cap F) \) then \( S \) is semi-open and hence \( S = V \cap cl S = V \cap cl U \) for some open set \( V \). If \( W = V \cup (X - cl U) \) then clearly \( A = W \cap cl A \). Thus \( A \in LC(X,T) \).

3. LC-CONTINUOUS FUNCTIONS.

In this section we define three distinct notions of LC-continuity and study some of their immediate consequences.

DEFINITION. A function \( f: X \rightarrow Y \) between spaces \( (X,T) \) and \( (Y,\sigma) \) is called

(i) LC-irresolute if \( f^{-1}(M) \in LC(X,T) \) for each \( M \in LC(Y,\sigma) \).

(ii) LC-continuous if \( f^{-1}(V) \in LC(X,T) \) for each \( V \in \sigma \).

(iii) sub-LC-continuous if there is a subbase (or, equivalently, a base) \( B \) for \( (Y,\sigma) \) such that \( f^{-1}(V) \in LC(X,T) \) for each \( V \in B \).

Let us note that these concepts have also obvious local forms. A function \( f: (X,T) \rightarrow (Y,\sigma) \) is called LC-irresolute [resp. LC-continuous] at a point \( x \in X \) if for each \( M \in LC(Y,\sigma) \) [resp. \( M \in \sigma \) ] satisfying \( f(x) \in M \) there is an open neighbourhood \( U \) of \( x \) in \( X \) such that \( U \cap cl f^{-1}(M) \subseteq f^{-1}(M) \). \( f: (X,T) \rightarrow (Y,\sigma) \) is said to be sub-LC-continuous at \( x \in X \) if there is an open subbase \( B \) for the neighbourhood filter of \( f(x) \) such that \( f^{-1}(V) \in LC(X,T) \) whenever \( V \in B \). It is easily verified that \( f: X \rightarrow Y \) is LC-irresolute [resp. LC-continuous, resp. sub-LC-continuous] if and only if it is LC irresolute [resp. LC-continuous, resp. sub-LC-continuous] at each point of \( X \).

From the previous definition it follows immediately that we have the following implications: continuous \( \Rightarrow \) LC-irresolute \( \Rightarrow \) LC-continuous \( \Rightarrow \) sub-LC-continuous. However, none of these implications can be reversed. Example 1 provides a function which is LC-irresolute but not continuous. In Example 2 we have constructed an LC-continuous function which is not LC-irresolute. Example 3 and Example 4 provide functions which are sub-LC-continuous but fail to be LC-continuous.

Our next two results are immediate consequences of Corollary 1 and Theorem 2 respectively.

PROPOSITION 6. A space \( (X,T) \) is submaximal if and only if every function having \( X \) as its domain is LC-continuous.

PROPOSITION 7. Let \( (X,T) \) be a space in which \( bd U \) is a discrete subset for each open set \( U \). Then for any space \( (Y,\sigma) \) and any LC-continuous function \( f: X \rightarrow Y \), \( f \) is LC-irresolute.

The importance of LC-continuous functions shows up in their relationship to nearly continuous functions. Recall that a function \( f: (X,T) \rightarrow (Y,\sigma) \) is said to be nearly continuous [7] if the inverse image of each open set is nearly open. The following result is an improvement of the decomposition theorem in [12].

THEOREM 3. A function \( f: X \rightarrow Y \) between spaces \( (X,T) \) and \( (Y,\sigma) \) is continuous if and only if \( f \) is nearly continuous and sub-LC-continuous.

PROOF. Let \( f \) be nearly continuous and sub-LC-continuous. Let \( B \) be a base for \( (Y,\sigma) \) such that \( f^{-1}(V) \in LC(X,T) \) whenever \( V \in B \). Now let \( W \in \sigma \) and \( f(x) \in W \). There is a \( V \in B \) such that \( f(x) \in V \subseteq W \). Since \( f^{-1}(V) \) is nearly open and locally closed it is
open \([12]\), hence \(x \in \text{int} f^{-1}(W)\). This proves the continuity of \(f\). The converse is obvious.

REMARK. Example 1 and Example 5 illustrate that near continuity and LC-continuity are independent of each other.

4. SOME PROPERTIES OF LC-CONTINUOUS FUNCTIONS.

It is obvious that if we consider the restriction of a function to an arbitrary subspace then LC-irresoluteness, LC-continuity or sub-LC-continuity are preserved. Example 3 illustrates that a function can be continuous on the elements of a cover of the domain but need not be LC-continuous. As an analogue to the case of continuous functions we have, however, the following result which is an immediate consequence of Theorem 1.

PROPOSITION 8. Let \(\{Z_i \mid i \in I\}\) be either an open or a locally finite closed cover of the space \((X,T)\). Let \(f: (X,T) \to (Y,\sigma)\) be such that \(f|_{Z_i}: Z_i \to Y\) is LC-irresolute [resp. LC-continuous, resp. sub-LC-continuous] for each \(i \in I\). Then \(f\) is LC-irresolute [resp. LC-continuous, resp. sub-LC-continuous].

Concerning compositions of functions, the composition of two LC-irresolute functions is clearly LC-irresolute. It is also easy to verify that whenever the composition of a continuous function and an LC-continuous function is defined, it is LC-continuous. In contrast to this we have the following two results.

PROPOSITION 9. The composition of an LC-continuous function and an LC-irresolute function need not be sub-LC-continuous.

PROOF. Let \(A = \{1/n \mid n \in \mathbb{N}\}\). But Corollary 2, \(A\) is locally closed in \(R\) and \(R-A\) is not. Let \(f: R \to R\) be as in Example 2, i.e. \(f(x) = x\) if \(x \in A\) and \(f(x) = 0\) if \(x \in R-A\). Then \(f\) is LC-continuous. Define \(g: R \to R\) be setting \(g(x) = 0\) if \(x < 0\) and \(g(x) = 1\) if \(x > 0\). Then \(g\) is clearly LC-irresolute. If \(h = gof\) then \(h(x) = 0\) if \(x \in R-A\) and \(h(x) = 1\) if \(x \in A\). Since the only possible preimages of sets under \(h\) are \(\emptyset, R, A\) and \(R-A\), \(h\) is not even sub-LC-continuous.

PROPOSITION 10. The composition of a sub-LC-continuous function and a continuous function need not be sub-LC-continuous.

PROOF. Take a sub-LC-continuous function \(f:(X,T) \to (Y,\sigma)\) which is not LC-continuous (e.g. the function in Example 3). Hence there is a set \(V \in \sigma\) such that \(f^{-1}(V) \notin \text{LC}(X,T)\). Now \(\sigma^* = \{\emptyset, V, Y\}\) is a topology on \(Y\) and the identity function \(id: (Y,\sigma) \to (Y,\sigma^*)\) is continuous. The composition \(idof\), however, fails to be sub-LC-continuous.

To every function \(f: X \to Y\) one can assign the graph function \(g_f: X \times X \times Y\) defined by \(g_f(x) = (x,f(x))\).

PROPOSITION 11. Let \(f: X \times Y\) be a function between spaces \((X,T)\) and \((Y,\sigma)\).

(i) If \(f\) is sub-LC-continuous then \(g_f\) is sub-LC-continuous.

(ii) If \(f\) is LC-irresolute then \(g_f\) need not be LC-continuous.

PROOF. Let \(B\) be a subbase for \((Y,\sigma)\) such that \(f^{-1}(V) \in \text{LC}(X,T)\) whenever \(V \in \sigma\). Then \(\{U : V \mid U \in T, V \notin B\}\) is a subbase for the product topology on \(X \times Y\). Since \(g_f^{-1}(U \times V) = U \cap f^{-1}(V)\), \(g_f\) is sub-LC-continuous. This proves (i). To prove (ii) let \(f: R \to \mathbb{R}\) as in Example 1, i.e. \(f(x) = 0\) if \(x < 0\) and \(f(x) = 1\) if \(x > 0\). Then \(B = \{(0,1]\} \cup \{1/n,1\} \mid n \in \mathbb{N}\}\) then \(B\) is closed in \(R \times R\). If \(V = R \times R - B\) then
Proposition 11 shows that the diagonal function of a family of LC-irresolute, thus LC-continuous, functions will not be LC-continuous in general. Our last result shows that sub-LC-continuous functions behave much better in this respect. Its proof is an immediate consequence of Proposition 5 and thus is omitted.

**PROPOSITION 12.** (1) For each \( i \in I \) let \( f_i: X_i \to Y_i \) be sub-LC-continuous. If \( f = \coprod f_i \) then \( f: \coprod X_i \to \coprod Y_i \) is sub-LC-continuous. (2) For each \( i \in I \) let \( f_i: X_i \to Y_i \) be sub-LC-continuous. If \( f = \Delta f_i: X \to \coprod Y_i \), i.e. \( (f(x))_i = f_i(x) \) for each \( i \in I \), then \( f \) is sub-LC-continuous.

5. EXAMPLES.

In this section we provide some examples in order to illustrate the various notions of generalized continuity which were introduced and discussed in the previous sections. We point out that in most cases we consider real-valued functions on the real line so that very natural spaces are involved in producing our counter examples.

**EXAMPLE 1.** Define a function \( f: \mathbb{R} \to \mathbb{R} \) by setting \( f(x) = x \) if \( x \leq 0 \) and \( f(x) = 1 \) if \( x > 0 \). For any subset \( V \subseteq \mathbb{R} \) we have \( f^{-1}(V) = V \cap (0, \infty) \) if \( \emptyset \neq V \) and \( \mathbb{R} \setminus V \) if \( V = \emptyset \). One easily checks that \( f \) is LC-irresolute. Obviously \( f \) is not continuous.

**EXAMPLE 2.** Let \( A = \{1/n \mid n \in \mathbb{N} \} \subseteq \mathbb{R} \). By Corollary 2, \( A \) is locally closed in \( \mathbb{R} \) and \( \mathbb{R} \setminus A \) is not. Define \( f: \mathbb{R} \to \mathbb{R} \) by setting \( f(x) = x \) if \( x \in A \) and \( f(x) = 0 \) if \( x \in \mathbb{R} \setminus A \). Since \( f^{-1}(\{0\}) = \mathbb{R} \setminus A \), \( f \) fails to be LC-irresolute. If \( V \subseteq \mathbb{R} \) is open and \( 0 \not\in V \) then \( f^{-1}(V) = \mathbb{R} \setminus V \) is locally closed by Proposition 2. If \( 0 \in V \) then \( f^{-1}(V) \) is a cofinite, hence an open subset of \( \mathbb{R} \). This shows that \( f \) is LC-irresolute. Obviously \( f \) is not continuous.

**EXAMPLE 3.** Define \( f: \mathbb{R} \to \mathbb{R} \) by setting \( f(x) = x \) if \( x \neq 0 \) and \( f(0) = 1 \). For any subset \( V \subseteq \mathbb{R} \) we have \( f^{-1}(V) = V \setminus \{0\} \) if \( \emptyset \neq V \) and \( f^{-1}(V) = V \cup \{0\} \) if \( V = \emptyset \). Hence, if \( V \) is an open interval then \( f^{-1}(V) \) is locally closed. Thus \( f \) is sub-LC-continuous. Now let \( V = \mathbb{R} \setminus (\{0\} \cup \{1/n \mid n \in \mathbb{N}, n > 2\}) \). Then \( V \) is open and dense. Since \( 1 \in V \) we have \((\mathfrak{cl} f^{-1}(V)) - f^{-1}(V) = \{x \in \mathbb{R} \mid x \neq 1/n \text{ for each } n \geq 2\} \) which is not closed, so \( f^{-1}(V) \) is not locally closed by Proposition 1. Hence \( f \) is not LC-continuous.

**EXAMPLE 4.** Let \( Y \) be the Sorgenfrey line and \( f: \mathbb{R} \to Y \) be the identity function. Clearly \( f \) is sub-LC-continuous. If \( B = \{-1/n \mid n \in \mathbb{N} \} \) then \( Y \setminus B \) is open in \( Y \) but not locally closed in \( \mathbb{R} \). Thus \( f \) is not LC-continuous.

**EXAMPLE 5.** There is a bijective, open and nearly continuous function \( f: \mathbb{R} \to Y \), \( Y \) a metrizable space, such that \( f \) is LC-continuous at no point.

Let \( D_1 \) be the set of all rationals in \( \mathbb{R} \) and let \( D_2 = \mathbb{R} - D_1 \). If \( Y = D_1 \cup D_2 \) then the identity function \( f: \mathbb{R} \to Y \), \( f(x) = x \) for each \( x \in \mathbb{R} \), is the desired function. This example is due to Berner [16].

**EXAMPLE 6.** There is a Hausdorff space \((X,\mathcal{T})\) and a bijective LC-irresolute function \( f: X \to Y \), \( Y \) a discrete space, such that \( f \) is continuous at no point.

We use the so-called Bourbaki construction. Let \( X \) be the set of reals and \( T_e \) be the euclidean topology on \( X \). Let \( \alpha \) be a maximal filter consisting of dense subsets of \((X,T_e)\) and let \( T = \alpha \) the topology on \( X \) having \( T_e \cup \alpha \) as a subbase. It is well known...
that \((X,T)\) is Hausdorff submaximal and dense-in-itself. If \(Y\) is the set of reals carrying the discrete topology then the identity function \(f: X \to Y\) is LC-irresolute since \((X,T)\) is submaximal. However, \(f\) is continuous at no point since \((X,T)\) is dense-in-itself.

ACKNOWLEDGEMENT. The first author wishes to acknowledge the support of an E. Schrödinger Auslandsstipendium awarded by Fonds zur Förderung der Wissenschaftlichen Forschung, Vienna, Austria.

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