ON α -CONVEX FUNCTIONS OF ORDER β OF RUSCHEWEYH TYPE

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Abstract: In this paper we consider functions $f(z) = z^p + a_{p+1}z^{p+1} + \dots$ which are regular in the unit disc $E = \{z : |z| < 1\}$ and satisfying the condition

$$\Re \left\{ (1-\alpha)(z^n f)^{(n+p)} / [(n+p)(z^{n-1} f)^{(n+p-1)}] \right. \\ \left. + \alpha(z^{n+1} f)^{(n+p+1)} / [(n+p+1)(z^n f)^{(n+p)}] \right\} > \beta.$$

We obtain a lower bound $\gamma(\alpha, \beta, n, p)$ such that

$$\Re \left\{ (z^n f)^{(n+p)} / [(n+p)(z^{n-1} f)^{(n+p-1)}] \right\} > \gamma(\alpha, \beta, n, p),$$

where p is a positive integer, n is any integer greater than $-p$, $\beta < 1$ and α is real.

KEY WORDS AND PHRASES α -convex functions of order β , Ruscheweyh type, p -valent functions, Hadamard product

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1. Introduction.

Let $A(p)$ be the class of functions $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$, p a positive integer,

which are regular in the unit disc E . S. RUSCHEWEYH [1] defined the class K_n consisting of all $f \in A(p)$ for $p = 1$ which satisfies

$$\Re \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > 1/2$$

for $z \in E$ and for some $n \in N_0 = N \cup \{0\}$, where

$$D^{n+p-1} f(z) = [z^p / (1-z)^{n+p}] * f(z)$$

with the operation $(*)$ stands for the Hadamard product, that is, if

$$g(z) = \sum_{k=0}^{\infty} a_k z^k, \quad h(z) = \sum_{k=0}^{\infty} b_k z^k, \quad \text{then } (g*h)(z) = \sum_{k=0}^{\infty} a_k b_k z^k; \text{ equivalently,}$$

$$D^{n+p-1} f(z) = [z^p (z^{n-1} f(z))^{(n+p-1)}] / (n+p-1)!$$

and showed that

$$K_{n+1} \subset K_n \quad (1.1)$$

for all $n \in N_0$. Familiar classes of K_n are $K_0 = S^*(1/2)$, $K_1 = K$, the starlike functions of order $1/2$ and the convex functions, respectively. In H. AL-AMIRI [2], H. AL-AMIRI combined two notions of RUSCHEWEYH and MOCANU [3] to introduce the Ruscheweyh-Mocanu α -convex functions of order n (denote it as $MR_n(\alpha)$) as follow: $f \in MR_n(\alpha)$ if

$$\operatorname{Re} \left\{ (1-\alpha) \frac{D^{n+1} f(z)}{D^n f(z)} + \alpha \frac{D^{n+2} f(z)}{D^{n+1} f(z)} \right\} > 1/2$$

$n \in N_0$, is satisfied for all $z \in E$ and for some real α . He showed that

$$MR_n(\alpha) \subset K_n \quad (1.2)$$

for $\alpha \geq 0$ and $n \in N_0$. Note that $MR_n(0) = K_n$, $MR_n(1) = K_{n+1}$. In GOEL and SOHI [4], GOEL and SOHI introduced the class

$$K_{n+p+1} = \{ f \in A(p) : \operatorname{Re} (D^{n+p} f(z) / D^{n+p-1} f(z)) > 1/2 \}$$

for $n \in N_0$ and $p \in N$ and proved the theorem:

$$K_{n+p} \subset K_{n+p-1} \quad (1.3)$$

In SONI [5], SONI had the generalization of R. SINGH and S. SINGH [6]:

$$R(n+p) \subset R(n+p-1) \quad (1.4)$$

where

$$R(n+p-1) = \{ f \in A(p) : \operatorname{Re} (D^{n+p} f(z) / D^{n+p-1} f(z)) > (n+p-1)/(n+p) \}.$$

In this paper we shall prove that if a function $f \in A(p)$ for some $p \in N$ and satisfies one of the conditions

$$\operatorname{Re} \left\{ (1-\alpha) \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} + \alpha \frac{D^{n+p+1} f(z)}{D^{n+p} f(z)} \right\} > \beta$$

for all $z \in E$, n is any integer greater than $-p$, then a lower bound $\gamma(\alpha, \beta, n, p)$ is obtained such that for all $z \in E$

$$\operatorname{Re} \left\{ \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} \right\} > \gamma(\alpha, \beta, n, p).$$

When $\beta = 1/2$, our Theorem yields

$$\gamma(\alpha, \beta, n, p) = 1/2 \quad \text{if } \alpha \geq -(n+p+1)$$

$$\gamma(\alpha, \beta, n, p) > 1/2 \quad \text{if } \alpha < -(n+p+1).$$

This implies (1.1) and (1.2) for $n \in N_0$, $\alpha = 1$ and for $n \in N_0$, $\alpha \geq 0$ respectively by taking $p = 1$. On the other hand, let $\alpha = 1$, $\beta = 1/2$ and $p \in N$, then

$$\gamma(\alpha, \beta, n, p) = 1/2$$

which is the result of (1.3). Moreover, if we take $\beta = (n+p)/(n+p+1)$, then

$$\gamma(\alpha, \beta, n, p) > (n+p-1)/(n+p)$$

for $0 < \alpha \leq 1$, which is (1.4) when $\alpha = 1$. Thus, our Theorem includes or improves all the results of (1.1), (1.2), (1.3) and (1.4).

2. Lemmas.

To prove the main result, we need the following lemmas

1. Jack's Lemma: Let $w(z)$ be regular in the unit disc E , with $w(0) = 0$. Then if $|w|$ attains its maximum value on the circle $|z| = r$ at a point z_1 , we can write

$$z_1 w'(z_1) = k w(z_1)$$

where $k \geq 1$. (JACK [7])

After some calculations we may obtain:

2. Lemma: For $p \in N$, n is any integer greater than $-p$ and α is real,

$$\begin{aligned} 0 \leq \{ & 2(n+p+1-\alpha) + 2\beta(n+p+1) - 3\alpha - [(2(n+p+1-\alpha) + 2\beta(n+p+1) - 3\alpha)^2 \\ & + 16(\alpha - \beta(n+p+1))(n+p+1-\alpha)]^{1/2} \} / [4(n+p+1-\alpha)] \leq 1/2 \\ & \text{when } \alpha/(n+p+1) \leq \beta \leq 1/2 \text{ and } n+p+1 > 2\alpha, \end{aligned} \quad (2.1)$$

$$\begin{aligned} 1/2 \leq \{ & 2\beta(n+p+1) - 3\alpha + [(2\beta(n+p+1) - 3\alpha)^2 \\ & + 8\alpha(n+p+1-\alpha)]^{1/2} \} / [4(n+p+1-\alpha)] < 1 \\ & \text{when } 1/2 \leq \beta < 1 \text{ and } \alpha \neq n+p+1, \end{aligned} \quad (2.2)$$

$$\begin{aligned} 1/2 \leq 1/(3-2\beta) < 1 \\ & \text{when } 1/2 \leq \beta < 1, \text{ and} \end{aligned} \quad (2.3)$$

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - n(D^{n+p-1}f(z)). \quad (2.4)$$

3. Main Result.

Theorem. If $f \in A(p)$ for some $p \in \mathbb{N}$ and satisfies the condition

$$\operatorname{Re} \left\{ (1-\alpha) \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} + \alpha \frac{D^{n+p+1} f(z)}{D^{n+p} f(z)} \right\} > \beta$$

for some n which is any integer greater than $-p$ and α is real, then

$$\operatorname{Re} \left\{ \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} \right\} > \gamma(\alpha, \beta, n, p)$$

where

$$\begin{aligned} \gamma(\alpha, \beta, n, p) = & \{ 2(n+p+1-\alpha) + 2\beta(n+p+1) - 3\alpha - [(2(n+p+1-\alpha) + 2\beta(n+p+1) - 3\alpha)^2 \\ & + 16(\alpha-\beta(n+p+1))(n+p+1-\alpha)]^{\frac{1}{2}} \} / [4(n+p+1-\alpha)] \\ & \text{if } \alpha/(n+p+1) \leq \beta < 1/2 \text{ and } n+p+1 > 2\alpha, \end{aligned}$$

$$\begin{aligned} \gamma(\alpha, \beta, n, p) = & \{ 2\beta(n+p+1) - 3\alpha + [(2\beta(n+p+1) - 3\alpha)^2 \\ & + 8\alpha(n+p+1-\alpha)]^{\frac{1}{2}} \} / [4(n+p+1-\alpha)] \\ & \text{if } 1/2 \leq \beta < 1 \text{ and } \alpha \neq n+p+1, \text{ and} \end{aligned}$$

$$\gamma(\alpha, \beta, n, p) = 1/(3-2\beta) \quad \text{if } 1/2 \leq \beta < 1 \text{ and } \alpha = n+p+1.$$

Proof: Suppose f satisfies the conditions in the theorem, and let $w(z)$ be a regular function such that

$$\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} = \frac{1 - (2\gamma - 1)w(z)}{1 - w(z)}, \quad (3.1)$$

where $\gamma = \gamma(\alpha, \beta, n, p)$, then $w(0) = 0$. The Theorem will follow if we can show that $|w(z)| < 1$ in E . Now by differentiating (3.1) logarithmically and applying (2.4), we get

$$\begin{aligned} & -(n+p) \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} + (n+p+1) \frac{D^{n+p+1} f(z)}{D^{n+p} f(z)} - 1 \\ &= \frac{2(1-\gamma)zw'(z)}{(1-w(z))(1-(2\gamma-1)w(z))} \end{aligned}$$

$$\begin{aligned} \text{Put } H(z) &= (1-\alpha) \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} + \alpha \frac{D^{n+p+1} f(z)}{D^{n+p} f(z)} - \beta \\ &= \frac{\alpha}{n+p+1} - \beta + \left(1 - \frac{\alpha}{n+p+1}\right) \frac{1 - (2\gamma - 1)w(z)}{1 - w(z)} \\ &\quad + \frac{2(1-\gamma)\alpha}{n+p+1} \frac{zw'(z)}{(1-w(z))(1-(2\gamma-1)w(z))}. \end{aligned}$$

If $|w| \not\prec 1$ in E , there exists, by Jack's Lemma, $z_1 \in E$ such that

$$|w(z_1)| = 1 \geq |w(z)| \text{ for all } |z| \leq |z_1| \text{ and } z_1 w'(z_1) = kw'(z_1) \text{ where } k \geq 1.$$

Let $w(z_1) = u + iv$. After simple computation, we get

$$\begin{aligned} & \operatorname{Re} \frac{z_1 w'(z_1)}{(1-w(z_1))(1-(2\gamma-1)w(z_1))} \\ &= \operatorname{Re} \frac{z_1 w'(z_1)}{w(z_1)} \cdot \frac{w(z_1)}{1-w(z_1)} \cdot \frac{1}{1-(2\gamma-1)w(z_1)} \\ & \quad \leq \gamma / [2(2\gamma^2 - 2\gamma + 1 - (2\gamma - 1)u)] \end{aligned}$$

$$\text{Let } g(u) = 1 / (2\gamma^2 - 2\gamma + 1 - (2\gamma - 1)u), \quad (3.2)$$

$$g'(u) = (2\gamma - 1) / (2\gamma^2 - 2\gamma + 1 - (2\gamma - 1)u)^2 \quad (3.3)$$

and $g(-1) = 1/(2\gamma^2)$, $g(1) = 1/(2(1-\gamma)^2)$.

From (2.1), $\alpha/(n+p+1) \leq \beta \leq 1/2$ and $n+p+1 > 2\alpha$ implies that

$$0 \leq \gamma(\alpha, \beta, n, p) = \gamma \leq 1/2, \text{ then } g'(u) \leq 0.$$

$$\text{Therefore we have } g(1) \leq g(u) \leq g(-1). \quad (3.4)$$

Taking the real part of $H(z_1)$ and applying (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned} \operatorname{Re} H(z) &= \frac{\alpha}{n+p+1} - \beta + (1 - \frac{\alpha}{n+p+1})\gamma - \frac{\gamma(1-\gamma)\alpha k}{n+p+1} g(u) \\ &\leq \frac{\alpha - \beta(n+p+1)}{n+p+1} + (1 - \frac{\alpha}{n+p+1})\gamma + \frac{-\gamma(1-\gamma)\alpha k}{n+p+1} \frac{1}{2(1-\gamma)^2} \\ &\leq -\{2(n+p+1-\alpha)\gamma^2 - [2(n+p+1-\alpha) + 2\beta(n+p+1) - 3\alpha]\gamma \\ &\quad - 2(\alpha - \beta(n+p+1))\} / [2(n+p+1)(1-\gamma)]. \end{aligned}$$

Since $\gamma(\alpha, \beta, n, p)$ is a root of the polynomial

$$2(n+p+1-\alpha)x^2 - [2(n+p+1-\alpha) + 2\beta(n+p+1) - 3\alpha]x - 2(\alpha - \beta(n+p+1)) = 0,$$

then $\operatorname{Re} H(z) \leq 0$, which contradicts to $\operatorname{Re} H(z) > 0$ for all $z \in E$. Thus,

$|w(z)| < 1$, for all $z \in E$. Therefore Theorem is proved in the case of $\alpha/(n+p+1) \leq \beta < 1/2$ and $n+p+1 > 2\alpha$. Similarly, the other cases of Theorem can be proved by using (2.2) and (2.3). Hence the proof is completed.

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