KELVIN PRINCIPLE FOR A CLASS OF SINGULAR EQUATIONS

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ABSTRACT. The classical Kelvin principle concerns invariance of solutions of the Laplace equation with respect to inversion in a sphere. By employing a hyperbolic-polar coordinate system, the principle is extended to cover a class of singular equations, which include the ultrahyperbolic equation.

KEY WORDS AND PHRASES. Kelvin principle, Laplace equation, ultrahyperbolic equation, Lorentzian distance.

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1. INTRODUCTION.

As is well known the classical Kelvin principle introduced in 1847 (Thomson [1]) concerns solutions of the Laplace equation. For solutions of some class of elliptic differential equations and their iterated forms in n independent variables, $n \geq 2$, the extension of Kelvin principle is usually proved using rectangular coordinates (Diaz and Martin [2], Germain and Bader [3], Huber [4], Weinstein [5]). In 1960 a generalization of Kelvin principle was established by Weinstein [5] for the equation

$$\sum_{i=1}^n \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial u}{\partial x_i} \right) = 0, \quad -\infty < k_i < \infty$$

using polar coordinates.

Following Weinstein method we shall give in this paper a new formulation of Kelvin principle for solutions of the class of singular partial differential equations

$$L(u) = \sum_{i=1}^n \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\alpha_i}{x_i} \frac{\partial u}{\partial x_i} \right) - \sum_{j=1}^m \left(\frac{\partial^2 u}{\partial y_j^2} + \frac{\beta_j}{y_j} \frac{\partial u}{\partial y_j} \right) + \frac{1}{r^2} P(u) = 0 \quad (1.1)$$

where α_i ($1 \leq i \leq n$) and β_j ($1 \leq j \leq m$) are real parameters, r is the Lorentzian metric defined by

$$r^2 = \sum_{i=1}^n x_i^2 - \sum_{j=1}^m y_j^2 \quad (1.2)$$

[illegible]

where r is the Lorentzian distance given by (1.2). We refer to this as "polar-hyperbolic transformation". In the polar-hyperbolic coordinate system, the operator (2.5) assumes the form

$$\square u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} - \sum_{j=1}^m \frac{\partial^2 u}{\partial y_j^2} = \frac{\partial^2 u}{\partial r^2} + \frac{n+m-1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \phi_2(u) \quad (2.7)$$

where ϕ_2 depends only on the variables $\phi_1, \dots, \phi_m, \psi_1, \dots, \psi_{n-1}$ and r is given in (1.2).

For example, the polar forms of Δu for $n = 2, 3$ are given by

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

$$\Delta u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} (u_{\theta\theta} + \frac{\cos \theta}{\sin \theta} u_\theta + \frac{1}{\sin^2 \theta} u_{\phi\phi})$$

The corresponding forms for the hyperbolic equations are given by

$$\Delta u = u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} u$$

$$\square u = u_{rr} + \frac{2}{r} u_r - \frac{1}{r^2} (u_{\theta\theta} + \frac{\sin\theta}{\cos\theta} u_\theta - \frac{1}{\cos^2\theta} u_{\phi\phi})$$

where $\square u = u_{xx} + u_{yy} - u_{zz}$, $r^2 = x^2 + y^2 - z^2$ and $x = r \operatorname{ch}\theta \cos\phi$, $y = r \operatorname{ch}\theta \sin\phi$, $z = r \operatorname{sh}\theta$.

3. A FUNDAMENTAL THEOREM AND MAIN RESULT.

In [5], using his main three recursion formulas, Weinstein gave the following theorem which will be used to establish our main result.

THEOREM 1. Let $v = v(r, \psi_1, \dots, \psi_{n-1})$ satisfy the differential equation

$$\frac{\partial^2 \mathbf{v}}{\partial r^2} + \frac{k}{r} \frac{\partial \mathbf{v}}{\partial r} = \frac{1}{r^2} \Phi(\mathbf{v}) \quad (3.1)$$

where k is a real or complex number and ϕ is a linear differential operator vanishing for $v = 0$ which is independent of the variable r . Then $\rho^{1-k} w$ satisfies the same

equation (3.1) in the variables $\rho, \psi_1, \dots, \psi_{n-1}$, where $\rho = 1/r$ and $w(\rho, \psi_1, \dots, \psi_{n-1}) = v(1/\rho, \psi_1, \dots, \psi_{n-1})$

Using Theorem 1 we can now establish an extension of Kelvin principle to ultrahyperbolic equations :

THEOREM 2. Let $u = u(x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_p)$ be a solution of the equation (1.1). Then

$$w = r^{-\lambda} u\left(-\frac{x_1}{r^2}, \dots, \frac{x_n}{r^2}, \frac{y_1}{r^2}, \dots, \frac{y_m}{r^2}, z_1, \dots, z_p\right) \quad (3.2)$$

is also a solution of the same equation (1.1), where

$$\lambda = n + m - 2 + \sum_{i=1}^n \alpha_i + \sum_{j=1}^m \beta_j \quad (3.3)$$

and r is the Lorentzian distance defined by (1.2).

PROOF. Let us consider the polar-hyperbolic transformation (2.6) which can be written in the following form

$$\begin{aligned} x_i &= r f_i(\phi, \psi), \quad i = 1, \dots, n \\ y_j &= r g_j(\phi, \psi), \quad j = 1, \dots, m \end{aligned} \quad (3.4)$$

where the notations $f_i(\phi, \psi)$, $g_j(\phi, \psi)$ or without subscripts $f(\phi, \psi)$, $g(\phi, \psi)$ denote functions of $\phi_1, \dots, \phi_m, \psi_1, \dots, \psi_{n-1}$.

We note that from (3.4) we have

$$\begin{aligned} \frac{\partial x_k}{\partial x_j} &= \frac{\partial x_k}{\partial r} \frac{\partial r}{\partial x_j} + \sum_{i=1}^m \frac{\partial x_k}{\partial \phi_i} \frac{\partial \phi_i}{\partial x_j} + \sum_{i=1}^{n-1} \frac{\partial x_k}{\partial \psi_i} \frac{\partial \psi_i}{\partial x_j} = \delta_{jk} \\ \frac{\partial y_k}{\partial y_j} &= \frac{\partial y_k}{\partial r} \frac{\partial r}{\partial y_j} + \sum_{i=1}^m \frac{\partial y_k}{\partial \phi_i} \frac{\partial \phi_i}{\partial y_j} = \delta_{jk} \end{aligned}$$

where δ_{jk} is the Kronecker delta. From (1.2) we have

$$\frac{\partial r}{\partial x_j} = \frac{\partial x_j}{\partial r} \quad (1 \leq j \leq n) \quad \text{and} \quad \frac{\partial r}{\partial y_j} = -\frac{y_j}{r} \quad (1 \leq j \leq m)$$

and we may express the partial derivatives $\partial \phi_i / \partial x_j$, $\partial \psi_i / \partial x_j$ and $\partial \phi_i / \partial y_j$ as a quotient the denominator of which is the Jacobian of the transformation (2.6). It can be shown that

$$\frac{\partial(x_1, \dots, x_n, y_1, \dots, y_m)}{\partial(r, \phi_1, \dots, \phi_m, \psi_1, \dots, \psi_{n-1})} = r^{n+m-1} h(\phi, \psi)$$

The numerator of this quotient contains obviously only a factor r^{n+m-2} , hence

$$\frac{\partial \phi_i}{\partial x_j} = \frac{1}{r} F_{ji}(\phi, \psi), \quad \frac{\partial \psi_i}{\partial x_j} = \frac{1}{r} G_{ji}(\phi, \psi), \quad \frac{\partial \phi_i}{\partial y_j} = \frac{1}{r} H_{ji}(\phi, \psi)$$

On the other hand, since

$$\begin{aligned} \frac{\partial u}{\partial x_j} &= \frac{x_j}{r} \frac{\partial u}{\partial r} + \frac{1}{r} \left(\sum_{i=1}^m \frac{\partial u}{\partial \phi_i} F_{ji}(\phi, \psi) + \sum_{i=1}^{n-1} \frac{\partial u}{\partial \psi_i} G_{ji}(\phi, \psi) \right) \\ \frac{\partial u}{\partial y_j} &= -\frac{y_j}{r} \frac{\partial u}{\partial r} + \frac{1}{r} \sum_{i=1}^m \frac{\partial u}{\partial \phi_i} H_{ji}(\phi, \psi) \end{aligned}$$

we see that

$$\frac{1}{x_j} \frac{\partial u}{\partial x_j} = \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \Psi_1(u), \quad 1 \leq j \leq n \quad (3.5)$$

$$\frac{1}{y_j} \frac{\partial u}{\partial y_j} = -\frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \Psi_2(u), \quad 1 \leq j \leq m$$

where the operators Ψ_1 and Ψ_2 depend only on the variables $\phi_1, \dots, \phi_m, \psi_1, \dots, \psi_{n-1}$.

If we substitute the expressions (2.7) and (3.5) into (1.1), then our equation (1.1) becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}(n+m-1 + \sum_{i=1}^n \alpha_i + \sum_{j=1}^m \beta_j) \frac{\partial u}{\partial r} + \frac{1}{r^2} \Psi(u) = 0 \quad (3.6)$$

Since $1-(n+m-1 + \sum_{i=1}^n \alpha_i + \sum_{j=1}^m \beta_j) = -\lambda$, by Theorem 1

$$w = r^{-\lambda} u\left(\frac{x_1}{r^2}, \dots, \frac{x_n}{r^2}, \frac{y_1}{r^2}, \dots, \frac{y_m}{r^2}, z_1, \dots, z_p\right)$$

satisfies the equation (1.1). Here we note that, since

$$\rho^2 = \sum_{i=1}^n (x_i/r^2)^2 + \sum_{j=1}^m (y_j/r^2)^2 = \frac{1}{r^2}$$

the substitution $\rho = 1/r$ in the solution u means replacing the variables x_i and y_j by x_i/r^2 and y_j/r^2 , respectively.

4. REMARKS.

(i) We note that, since the r defined by (1.2) is not real for $\sum_{i=1}^n x_i^2 < \sum_{j=1}^m y_j^2$, the solutions of (1.1) is valid only in the domain $D \times \Omega$, where

$$D = D_n \times D_m = \{(x, y) : x \in D_n, y \in D_m, \sum_{i=1}^n x_i^2 > \sum_{j=1}^m y_j^2\}$$

is a hyperconoidal domain in R^{n+m} . Here D_n and D_m are the spherical domains centered at the origins of R^n and R^m , respectively, and $\Omega \subset R^p$ is the regularity domain of u with respect to the variable z .

(ii) In equation (1.1) if we have addition instead of subtraction of the two summations, then Theorem 2 remains valid, where

$$r^2 = \sum_{i=1}^n x_i^2 + \sum_{j=1}^m y_j^2$$

This includes Weinstein's [5] and Altin's [2] results.

(iii) In the special case $Pu = Yu$ where $Y = \text{const.}$ the formula (3.2) gives the result obtained in [2].

(iv) If we multiply both sides of the equation (1.1) by -1 , we get the equation

$$-L(u) = \sum_{j=1}^m \left(\frac{\partial^2 u}{\partial y_j^2} + \frac{\beta_j}{y_j} \frac{\partial u}{\partial y_j} \right) - \sum_{j=1}^n \left(\frac{\partial^2 u}{\partial x_j^2} + \frac{\alpha_j}{x_j} \frac{\partial u}{\partial x_j} \right) + \frac{1}{r^2} P(u) = 0$$

where $r_1^2 = \sum_{j=1}^m y_j^2 - \sum_{j=1}^n x_j^2 = -r^2$. This shows that if $u(x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_p)$ is a solution of the equation (1.1), then by Theorem 2

$$w_1 = r_1^{-\lambda} u\left(\frac{x_1}{r^2}, \dots, \frac{x_n}{r^2}, \frac{y_1}{r^2}, \dots, \frac{y_m}{r^2}, z_1, \dots, z_p\right)$$

is also a solution of the same equation (1.1), where λ is given by (3.3). It is clear that this solution is valid in a different domain where $r_1^2 > 0$, that is, $r^2 < 0$.

(v) It is clear that by a simple linear transformation, Theorem 2 also holds for a more general equation of the form

$$\sum_{i=1}^n a_i^2 \left(\frac{\partial^2 u}{\partial \zeta_i^2} + \frac{\alpha_i}{\zeta_i - \zeta_i^0} \frac{\partial u}{\partial \zeta_i} \right) - \sum_{j=1}^m b_j^2 \left(\frac{\partial^2 u}{\partial \eta_j^2} + \frac{\beta_j}{\eta_j - \eta_j^0} \frac{\partial u}{\partial \eta_j} \right) + \frac{1}{r^2} P(u) = 0$$

where $a_i, b_i, \alpha_i, \beta_i$ are real parameters ($a_i \neq 0, b_i \neq 0$), $\zeta^0 = (\zeta_1^0, \dots, \zeta_n^0)$ and $\eta^0 = (\eta_1^0, \dots, \eta_m^0)$ are fixed points in D_n and D_m , respectively. Here

$$w = r^{-\lambda} u\left(\frac{\zeta_1 - \zeta_1^0}{a_1 r^2}, \dots, \frac{\zeta_n - \zeta_n^0}{a_n r^2}, \frac{\eta_1 - \eta_1^0}{b_1 r^2}, \dots, \frac{\eta_m - \eta_m^0}{b_m r^2}, z_1, \dots, z_p\right)$$

where

$$r^2 = \sum_{i=1}^n \left(\frac{\zeta_i - \zeta_i^0}{a_i} \right)^2 - \sum_{j=1}^m \left(\frac{\eta_j - \eta_j^0}{b_j} \right)^2$$

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