SUFFICIENT CONDITIONS FOR SPIRAL-LIKENESS

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ABSTRACT. Coefficient conditions sufficient for spiral-likeness are found by convolution methods. The order of starlikeness for such functions is also determined.

KEY WORDS AND PHRASES. Spiral-like, starlike, convolution.

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INTRODUCTION.

A function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) analytic in the unit disk \( \Delta = \{ |z| < 1 \} \) is said to be starlike of order \( \alpha \), \( 0 < \alpha < 1 \), if \( \Re \{ \frac{zf'}{f} \} > \alpha \), \( z \in \Delta \), and is said to be spiral-like, \( -\pi/2 < \lambda < \pi/2 \), if \( \Re \{ e^{i\lambda} \frac{zf'}{f} \} > 0 \), \( z \in \Delta \). We denote these classes, respectively, by \( S^*(\alpha) \) and \( S^{p}(\lambda) \). Note that \( S^*(0) = S^{p}(0) \), the family of starlike functions. Functions in \( S^{p}(\lambda) \) were shown by Spacek [4] to be univalent in \( \Delta \) and were later studied extensively by Libera [1].

A function \( f \) is in \( S^*(\alpha) \) if its coefficients are sufficiently small.

THEOREM A [2]. If the coefficients of \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) satisfy the inequality

\[
\frac{\pi}{n^2} \frac{n-\alpha}{1-\alpha} |a_n| < 1,
\]

then \( \left| \frac{zf'/f - 1}{\pi} \right| < 1 - \alpha \), \( z \in \Delta \), and hence \( f \in S^*(\alpha) \).

It is our purpose here to find coefficient conditions guaranteeing that \( f \) is in \( S^*(\alpha) \). Our methods will involve convolution properties and will also furnish us with an alternate proof that (1) is a sufficient condition for \( f \) to be in \( S^*(\alpha) \).

The convolution or Hadamard product of two power series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \)

and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) is defined as the power series \( (f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \). In [3] it is shown that \( f \in S^*(\alpha) \) if and only if

\[
\frac{1}{z} (f \ast \frac{z + ((x+2\alpha-1)/(2-2\alpha))z^2}{(1-z)^2}) \neq 0 \quad (z \in \Delta, \; |x| = 1)
\]

and \( f \in S^{p}(\lambda) \) if and only if

\[
\frac{1}{z} (f \ast \frac{z + ((x-e^{-2\lambda})/(1+e^{-2\lambda}))z^2}{(1-z)^2}) \neq 0 \quad (z \in \Delta, \; |x| = 1).
\]

Now \( \frac{z + cz^2}{(1-z)^2} = \sum_{n=2}^{\infty} \frac{n(n+1)c}{n^2} (n+1)z^n \), so we may restate these results as
THEOREM B. The function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(a) \) (Sp(\( \lambda \))) if and only if

\[
1 + \sum_{n=2}^{\infty} (n+1)c_n z^{n-1} \leq 0 \quad \text{for all} \quad z \in \Delta \quad \text{and} \quad |x| = 1, \quad \text{where}
\]

\[
c = \frac{x+2a-1}{2-2a} \left( c = \frac{x-e^{-21\lambda}}{1+e^{-21\lambda}} \right).
\]

Since \(|1 + \sum_{n=2}^{\infty} (n+1)c_n z^{n-1}| > 1 - \sum_{n=2}^{\infty} |n+1c||a_n||z|^{n-1}|\),

we conclude from Theorem B that condition (1) is sufficient for \( f \) to be in \( S^*(a) \) or \( Sp(\lambda) \) is that \( \sum_{n=2}^{\infty} |n+1c||a_n| < 1 \) for the appropriate choice of \( c \). A straightforward computation shows that

\[
\left| n + \frac{(n-1)x+2a-1}{2-2a} \right| < \left| n + \frac{(n-1)(2a)}{2-2a} \right| = \frac{n-a}{1-a},
\]

and we can conclude from Theorem B that condition (1) is sufficient for \( f \) to be in \( S^*(a) \). The corresponding result for \( Sp(\lambda) \) is computationally more involved.

2. THE MAIN CLASS.

THEOREM 1. The function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Sp(\lambda) \) if \( \sum_{n=2}^{\infty} B_n(\lambda)|a_n| < 1 \) for

\[
B_n(\lambda) = \sqrt{(n-1)^2 + 4n \cos^2 \lambda}.
\]

The result is sharp, with \( f_n(z) = z + a_n z^n \in Sp(\lambda) \) if and only if \( |a_n| < 1/B_n(\lambda) \).

PROOF. From Theorem B, it suffices to show for \( c = (x-e^{-21\lambda})/(1+e^{-21\lambda}) \) that

\[
\max_{|x|=1} \left| n + (n-1)c \right| = B_n(\lambda). \quad \text{Writing} \quad c = c_1 + ic_2, \quad c_1 \text{ and } c_2 \text{ real, we have}
\]

\[
(2) \quad \left| n + (n-1)c \right| = \sqrt{n^2 + (n-1)((c_1^2 + c_2^2) + 2nc_1)}.
\]

Hence \( n + (n-1)c \) will attain its maximum when \( (n-1)(c_1^2 + c_2^2 + 2nc_1) \) does. Setting \( x = e^{i\beta} \), a computation shows that

\[
c_1^2 + c_2^2 = \frac{1 - \cos(2\lambda + \beta)}{1 + \cos 2\lambda} \quad \text{and} \quad c_1 = \frac{\cos(2\lambda + \beta) + \cos \beta - (1 + \cos 2\lambda)}{2(1 + \cos 2\lambda)}.
\]

Thus

\[
(n-1)(c_1^2 + c_2^2 + 2nc_1) = \frac{\cos(2\lambda + \beta) + n \cos \beta - (1 + n \cos 2\lambda)}{2(1 + \cos 2\lambda)},
\]

which is maximized when \( g(\beta) = \cos(2\lambda + \beta) + n \cos \beta \) is maximized. But \( g(\beta) \) attains its maximum when \( \beta = \beta_0 = \tan^{-1}\left(-\frac{\sin 2\lambda}{n + \cos 2\lambda}\right) \), with \( g(\beta_0) = \sqrt{(n-1)^2 + 4n \cos^2 \lambda} \). For this choice of \( \beta_0 \), we have

\[
(n-1)(c_1^2 + c_2^2 + 2nc_1) = \frac{\sqrt{(n-1)^2 + 4n \cos^2 \lambda} + (n-1) - 2n \cos 2\lambda}{2 \cos 2\lambda} = t_n(\lambda),
\]
It now follows from (2) that
\[
\max_{|z|=1} \left| n + (n-1)c \right| = \left( n^2 + (n-1)t_n^2(\lambda) \right)^{1/2},
\]
which may be expressed as \( B(\lambda) \). This completes the proof. To show sharpness, note that according to Theorem \( B \) \( f_n(z) = z + a_n z^n \) \( \in \mathcal{S}(\lambda) \) if \( z^{-1} = -\left( n + (n-1)c \right) a_n^{-1} \)
has a solution for \( z \in \Delta \). Choosing \( c \) so that \( |n + (n-1)c| = B_n(\lambda) \), we see that
\[
f_n(z) \notin \mathcal{S}(\lambda) \text{ if } |a_n| > 1/B_n(\lambda).
\]
In particular, \( f_n(z) = z + a_n z^n \in \mathcal{S}(\lambda) \) if and only if \( |a_n| < 1/B_n(\lambda) \).

**COROLLARY 1.** If \( f_n(z) = z + a_n z^n \in \mathcal{S}\left(\lambda_0^0\right) \), then \( f_n \in \mathcal{S}(\lambda) \) for \( |\lambda| < |\lambda_0| \).

**PROOF.** This is a consequence of \( B_n(\lambda) \) being an increasing function of \( |\lambda| \).

In fact, any function that satisfies the conditions of Theorem 1 for \( \lambda = \lambda_0 \) will also be in \( \mathcal{S}(\lambda) \) for \( |\lambda| < |\lambda_0| \), a sharp contrast to the inclusion properties for the general class \( \mathcal{S}(\lambda) \). The function
\[
f_\lambda(z) = z(1-z)^{-1}e^{-1} \cos \lambda \text{ is in } \mathcal{S}(\lambda) \text{ but it is not in } \mathcal{S}(\lambda) \text{ for any } \gamma \neq \lambda.
\]
On the other hand, the upper bound on the modulus of the coefficients for \( f \in \mathcal{S}(\lambda) \) is a decreasing function of \( |\lambda| \). Zamorski [5] showed the sharp coefficient bounds \( |a_n| \) for \( f \in \mathcal{S}(\lambda) \) to be
\[
|a_n| = \prod_{k=1}^{n-1} \sqrt{(k-1)^2 + 4k \cos^2 \lambda/(n-1)!}, \text{ with } f(z) \text{ being extremal.}
\]

Though Theorem 1 gives a sharp result, it is not aesthetically pleasing because of the complicated nature of \( B_n(\lambda) \). A consequence of the inequality
\[
1 + (n-1)\sec \lambda > B_n(\lambda)
\]
is more palatable sufficient condition.

**COROLLARY 2.** If \( \sum_{n=2}^{\infty} (1 + (n-1)\sec \lambda)|a_n| < 1 \), then \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}(\lambda) \).

Using a different method, Corollary 2 will also be shown to follow from Theorem 3.

3. ORDER OF STARRLIKEHNESS.

Since \( B_n(\lambda) > n \), we see from Theorem A that a function satisfying the conditions of Theorem 1 must be starlike. We can actually do better.

**THEOREM 2.** If \( f \) satisfies the conditions of Theorem 1 then \( f \in S^*(\alpha_0) \) for \( \alpha_0 = (3 - \sqrt{1 + 8 \cos^2 \lambda})/2(1 + \cos \lambda) \).

The result is sharp, with extremal function
\[
f_2(z) = z + z^2/B_2(\lambda).
\]

**PROOF.** In view of Theorem A, we need only show that \( B_n(\lambda) > (n-\alpha_0)/(1-\alpha_0) \) for every \( n \). Since \( B_2(\lambda) = (2-\alpha_0)/(1-\alpha_0^0) \), it suffices to prove that \( (1-\alpha_0)B_n(\lambda)/(n-\alpha_0) \) is an increasing function of \( n \). Setting
\[
G(x) = x^{-1} + \sqrt{(x-1)^2 + x \cos^2 \lambda}/x^\alpha_0
\]
I will show that \( G'(x) > 0 \) for \( x > 2 \).
A differentiation of $G$ leads to

$$(x-a_0)^2 G'(x) = 1-a_0 + \frac{(1-a_0-2\cos^2\lambda)x - (1-a_0+2a_0\cos^2\lambda)}{\sqrt{(x-1)^2 + 4x\cos^2\lambda}}$$

$= H(x)$, say.

Since $H'(x) = \frac{4\sin^2\lambda\cos^2\lambda(x-a_0)}{((x-1)^2 + 4x\cos^2\lambda)^{3/2}} > 0$, it follows for $x > 2$ that

$H(x) > H(2) \cos (1 - \frac{4\cos \lambda - 1}{\sqrt{1 + 8\cos^2\lambda}}) > 0.$

Therefore $G(x)$, and consequently $(1-a_0)B_n(\lambda)/(n-a_0)$, is an increasing function. This completes the proof.

We have actually shown more according to Theorem A.

COROLLARY. If $f$ satisfies the conditions of Theorem 1, then

$$|zf'/f| - 1 < 1-a_0, z \in \Delta.$$  

Functions in $S^*(a)$ need not be in $Sp(\lambda)$ for $\lambda \neq 0$. The function $f_a(z) = z/(1-z)^{2(1-a)} \in S^*(a)$ since $zf'/f$ maps $\Delta$ onto the half plane $Re \ w > a$. However $f_a \not\in Sp(\lambda), \lambda \neq 0$. We next look at a subclass of $S^*(a)$ whose functions are spiral-like.

THEOREM 3. If $f(z) = z + \ldots$ is analytic with

$$|(zf'/f) - 1| < 1-a \text{ for } z \in \Delta,$$

then $f \in Sp(\lambda)$ for $|\lambda| < \cos^{-1}(1-a)$. The result is sharp, with extremal function $f(z) = z(1-a)z$.

PROOF. We may write $(zf'/f) - 1 = (1-a)\omega(z)$, where $|\omega(z)| < 1$ for $z \in \Delta$. Thus $Re\{e^{i\lambda}zf'/f\} = \cos \lambda + (1-a) Re\{e^{i\lambda}\omega(z)\} > \cos \lambda - (1-a) > 0$ for $|\lambda| < \cos^{-1}(1-a)$, and the proof is complete.

COROLLARY. If $|zf'/f - 1| < \cos \lambda$, then $f \in Sp(\lambda)$.

PROOF. Set $a = 1 - \cos \lambda$ in Theorem 3.

Finally an application of Theorem A, with $a = 1 - \cos \lambda$, to Theorem 3 provides us with an alternate proof to Corollary 2 of Theorem 1.

REFERENCES

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