VISCOSITY SOLUTIONS ASSOCIATED WITH IMPULSE CONTROL PROBLEMS FOR PIECEWISE-DETERMINISTIC PROCESSES

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ABSTRACT. This paper considers existence and uniqueness results for viscosity solutions of integro-differential equations associated with the impulse control problem for piecewise-deterministic processes on bounded domains and on $\mathbf{R}^n$.

KEY WORDS AND PHRASES. Impulse control, viscosity solution, piecewise-deterministic processes.

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1. INTRODUCTION.

This paper considers viscosity solutions of integro-differential equations associated with the impulse control problem for piecewise-deterministic (PD) processes

$$\max(Lu - f, u - Mu) = 0 \text{ in } E \tag{1.1}$$

where

$$Lu(x) = -\sum_{i=1}^{n} g_i(x)u_{x_i}(x) + \alpha(x)u(x) - \lambda(x)\int_E (u(y) - u(x))\, Q(dy, x)$$
$$Mu(x) = \inf_{\substack{x+\xi\epsilon\overline{E} \\ \xi \geq 0}} (u(x + \xi) + c(\xi)).$$

We consider the cases when $E = \Omega$, a bounded domain in $\mathbf{R}^n$, and $E = \mathbf{R}^n$. In the bounded domain case, we have the following boundary condition:

$$u(x) = \int_\Omega u(y)Q(dy, x) \text{ for all } x\epsilon\partial\Omega. \qquad (1.2)$$

Let us briefly give the background for this problem. A PD process, $x(t)$, with jump rate, $\lambda(x)$, and jump distribution $Q(dy, x)$, follows deterministic dynamics between random jumps:

$$\frac{dx(t)}{dt} = (g_1(x(t)), \ldots, g_n(x(t))).$$

Davis [1] developed the probabilistic side of these PD Markov processes. If the ith jump of the process occurs at T_i, then the distribution of $x(T_i)$ is $Q(dy, x(T_i^-))$ and

$$P(T_{i+1} - T_i > t) = \exp\left(-\int_0^t \lambda(x(T_i + s))\,ds\right).$$

Davis [1] showed that a PD process is a strong Markov process with generator

$$\sum_{i=1}^n g_i(x)u_{x_i} + \lambda(x)\int_E (u(y) - u(x))Q(dy, x),$$

with E, the state space. The boundary condition occurs because the PD process jumps back into the interior of Ω, upon hitting the boundary of Ω. The jumps T_i are part of the process, $x(t)$. Consider when certain jumps, "impulses", are controlled from outside the PD process. Suppose the state is changed from x to $x + \xi$ with impulse $\xi \geq 0$, and cost $\ell(\xi)$ is incurred when the impulse ξ is applied. An impulse control strategy v is a sequence of stopping times and impulses,

$$v = \{\theta_1, \xi_1, \theta_2, \xi_2, \ldots\}, \qquad (\theta_i \to \infty \text{ as } i \to \infty).$$

The controlled PD process x^v satisfies

$$x^v(\theta_i + 0) = x^v(\theta_i - 0) + \xi_i. \qquad (1.3)$$

The associated cost function is

$$J_x(v) = E_x\left[\int_0^\infty f(x^v(t))e^{-\alpha t}dt + \sum_{i=1}^\infty c(\xi_i)e^{-\alpha\theta_i}\right]$$

The minimal cost function is

$$V(x) = \inf_v J_x(v). \qquad (1.4)$$

Heuristically, the dynamic programming equation satisfied by the minimal cost function is given by (1.1).

For results on optimal control of PD processes, see Davis [2], Vermes [3], Soner [4], Lenhart and Liao [5, 6], and Gugerli [7]. See Barles [8] for deterministic impulse control.

In this paper, we define the notion of viscosity solution of (1.1) and prove existence and uniqueness results in the viscosity sense. The control representation is discussed in the last section.

2. UNIQUENESS.

The original formulations of viscosity solution definitions, by Crandall and Lions [9, 10], did not include integral terms in the operators, so we state the definition extension to this case.

DEFINITION: $u \epsilon BUC(\overline{E})$ (bounded, uniformly continuous) (E will be Ω or $\mathbf{R}^n$)

(i) u is a *viscosity subsolution* of (1.1) if

$$\max \Big(-g_i(x_0)\phi_{x_i}(x_0) + \alpha(x_0)u(x_0)$$

$$- \lambda(x_0) \int_E (u(y) - u(x_0))\, Q(dy, x_0) - f(x_0),$$

$$u(x_0) - Mu(x_0)\Big) \leq 0 \quad (2.1)$$

whenever $\phi \epsilon C^1(E)$ and $u - \phi$ has a global maximum at x_0.

(ii) u is a *viscosity supersolution* of (1.1) if

$$\max \Big(-g_i(x_0)\phi_{x_i}(x_0) + \alpha(x_0)u(x_0)$$

$$- \lambda(x_0) \int_E (u(y) - u(x_0))\, Q(dy, x_0) - f(x_0),$$

$$u(x_0) - Mu(x_0)\Big) \geq 0 \quad (2.2)$$

whenever $\phi \epsilon C^1(E)$ and $u - \phi$ has a global minimum at x_0.

Note that implicit summation on repeated subscripts is used on the g_i terms above.

We make the following assumptions:

$$\lambda, f, \alpha \text{ bounded uniformly continuous on } E, \quad (2.3)$$

$$g_i \text{ Lipschitz continuous on } E, \ i = 1, \ldots, n, \quad (2.4)$$

$$\alpha(x) \geq \alpha_0 > 0 \text{ on } E, \quad (2.5)$$

$$\lambda(x) \geq 0, \ f(x) \geq 0 \text{ on } E, \quad (2.6)$$

for each fixed $x \epsilon E$, $Q(dy, x)$ is a probability measure which is Lipschitz continuous as a function of x, i.e.,

$$\left| \int_E \phi(y) Q(dy, x) - \int_E \phi(y) Q(dy, z) \right| \leq C_Q \|\phi\|_{L^\infty} |x - z| \text{ for all } \phi \epsilon L^\infty(E), \quad (2.7)$$

and $c(\cdot)$ is a continuous, subadditive, increasing, positive function on $(\mathbf{R}^+)^n$ with

$$c(\xi) \to \infty \text{ as } \xi \to \infty, \tag{2.8}$$

$$c(0) = k,$$

$$c(\xi) \geq k > 0 \text{ for all } \xi \geq 0. \tag{2.9}$$

We assume that Ω is a smooth bounded domain in $\mathbf{R}^n$.

We need an assumption that guarantees

$$u \epsilon C(\overline{\Omega}) \Rightarrow Mu \epsilon C(\overline{\Omega}),$$

from Lions and Perthame [11], we have the needed assumption:

$$\text{for all } x \epsilon \overline{\Omega}, \quad \{\xi \geq 0 \mid \xi \neq 0, \ x + \xi \epsilon \partial\Omega, \ \exists \varepsilon > 0, \ \text{s.t. } \forall y \geq 0,$$

$$x + y \notin \Omega \text{ if } |y - \xi| \leq \varepsilon\} \text{ is empty.} \tag{2.10}$$

If Ω is convex, then (2.10) holds.

We now prove comparison results in Ω and then in $\mathbf{R}^n$ that yield uniqueness results for equation (1.1).

THEOREM 2.1. *Under assumptions (2.3)–(2.10) on Ω, if u is a viscosity subsolution of (1.1) on Ω and v is a viscosity supersolution of (1.1) on Ω with*

$$u(x) \leq \int_\Omega u(y) Q(dy, x) \ \text{ and } \ v(x) \geq \int_\Omega v(y) Q(dy, x) \ \text{ for all } \ x \in \partial\Omega, \tag{2.11}$$

then $u \leq v$ on $\overline{\Omega}$.

PROOF: Let $0 < \mu < 1$ and set $w = \mu u$. There exists z in $\overline{\Omega}$ such that

$$(w - v)(z) = \max_{x \epsilon \overline{\Omega}} (w - v)(x).$$

First we show that we can choose z so that $z \notin \partial\Omega$. If $z \in \partial\Omega$, then

$$\max(w - v) = w(z) - v(z) \leq \int_\Omega (w(y) - v(y)) \, Q(dy, z) \text{ by (2.11)}$$

$$\leq \sup(w - v).$$

If the maximum of $w - v$ does not occur at an interior point, then there exists a set $A \subset \Omega$ such that

$$Q(A, z) > 0 \text{ and } w(y) - v(y) < \max(w - v) \text{ for all } y \epsilon A.$$

Then

$$\int_\Omega (w(y) - v(y)) \, Q(dy, z) < \sup(w - v),$$

which is a contradiction. So there exists $\overline{z} \in \Omega$ where the maximum occurs. Thus we can assume $z \in \Omega$.

Set $M = \max(\|u\|_\infty, \|v\|_\infty)$. Define

$$\Phi_\epsilon(x, y) = w(x) - v(y) - \left|\frac{x - y}{\epsilon}\right|^2 - C_\epsilon |y - z|^2$$

where $C_\epsilon = \sqrt{\omega_v\left(\sqrt{2M\epsilon}\right)}$ and ω_v is a modulus of continuity for v. There exists $(x^\epsilon, y^\epsilon) \in \overline{\Omega}$ such that

$$\sup_{x, y \in \overline{\Omega}} \Phi_\epsilon(x, y) = \Phi_\epsilon(x^\epsilon, y^\epsilon).$$

Then

$$\Phi_\epsilon(z, z) = w(z) - v(z) \leq \Phi_\epsilon(x^\epsilon, y^\epsilon)$$

implies

$$\left|\frac{x^\epsilon - y^\epsilon}{\epsilon}\right|^2 + C_\epsilon |y^\epsilon - z|^2 \leq w(x^\epsilon) - v(x^\epsilon) - (w(z) - v(z)) + v(x^\epsilon) - v(y^\epsilon)$$

$$\leq 2M.$$

Refining this estimate gives

$$|x^\epsilon - y^\epsilon| \leq \sqrt{2M}\epsilon$$

and then

$$\left|\frac{x^\epsilon - y^\epsilon}{\epsilon}\right|^2 + C_\epsilon |y^\epsilon - z|^2 \leq \omega_v\left(|x^\epsilon - y^\epsilon|\right)$$

$$\leq \omega_v\left(\sqrt{2M}\epsilon\right).$$

This implies

$$\left|\frac{x^\epsilon - y^\epsilon}{\epsilon}\right| \to 0 \text{ as } \epsilon \to 0. \tag{2.12}$$

Also

$$C_\epsilon |y^\epsilon - z|^2 \leq C_\epsilon^2,$$

and then

$$|y^\epsilon - z| \to 0 \text{ as } \epsilon \to 0. \tag{2.13}$$

Notice also we have for ϵ sufficiently small, $x^\epsilon, y^\epsilon \in \Omega$ since $z \in \Omega$.

Since w is a viscosity subsolution of

$$\max\left(-g_i w_{x_i} + \alpha w - \lambda \int_\Omega (w(y) - w(x)) Q(dy, x) - \mu f, w - \widetilde{M}w\right) = 0,$$

$$\widetilde{M}w(x) = \inf_{\substack{x + \xi \in \overline{\Omega} \\ \xi \geq 0}} (w(x + \xi) + \mu C(\xi))$$

and $x \to w(x) - \left(v(y) - \left|\frac{x - y^\epsilon}{\epsilon}\right|^2 - C_\epsilon |y^\epsilon - z|^2\right)$ has a maximum at x^ϵ,

$$\max\left(-g_i(x^\varepsilon)\left(2\left(\frac{x^\varepsilon - y^\varepsilon}{\varepsilon^2}\right)_i\right)\right.$$

$$+\, \alpha(x^\varepsilon)w(x^\varepsilon) - \lambda(x^\varepsilon)\int_\Omega (w(y) - w(x^\varepsilon))\, Q(dy, x^\varepsilon) - \mu f(x^\varepsilon),$$

$$\left. w(x^\varepsilon) - \widetilde{M}w(x^\varepsilon)\right) \le 0. \quad (2.14)$$

Since v is a supersolution of (1.1) and

$$y \to v(y) - \left(w(x_0) - \left|\frac{x_0 - y}{\varepsilon}\right|^2 - C_\epsilon\, |y_0 - z|^2\right),$$

has a minimum at y,

$$\max\left(-g_i(y^\varepsilon)\left(2\left(\frac{x^\varepsilon - y^\varepsilon}{\varepsilon^2}\right)_i + 2C_\epsilon\, (y^\varepsilon - z)_i\right)\right.$$

$$+\, \alpha(y^\varepsilon)v(y^\varepsilon) - \lambda(y^\varepsilon)\int_\Omega (v(y) - v(y^\varepsilon))\, Q(dy, y^\varepsilon) - f(y^\varepsilon),$$

$$\left. v(y^\varepsilon) - Mv(y^\varepsilon)\right) \ge 0. \quad (2.15)$$

CASE A. $v(y^\varepsilon) \ge Mv(y^\varepsilon)$.

There exists $\xi^1 \ge 0$ such that $Mv(x^\varepsilon) = v(x^\varepsilon + \xi^1) + c(\xi^1)$. From property (2.10), $Mv \in C(\overline{\Omega})$, and $Mv(x^\varepsilon) - Mv(y^\varepsilon)$ can be made small for ε small.

$$\Phi_\varepsilon(x^\varepsilon, y^\varepsilon) \le w(x^\varepsilon) - Mv(y^\varepsilon) - \left|\frac{x^\varepsilon - y^\varepsilon}{\varepsilon}\right|^2 - C_\epsilon\, |y^\varepsilon - z|^2$$

$$\le w(x^\varepsilon) - Mv(x^\varepsilon) - \left|\frac{x^\varepsilon - y^\varepsilon}{\varepsilon}\right|^2 - C_\epsilon\, |y^\varepsilon - z|^2 + Mv(x^\varepsilon) - Mv(y^\varepsilon)$$

$$\le w(x^\varepsilon + \xi^1) + \mu c(\xi^1) - \left(v(x^\varepsilon + \xi^1) + c(\xi^1)\right)$$

$$- \left|\frac{x^\varepsilon - y^\varepsilon}{\varepsilon}\right|^2 - C_\epsilon\, |y^\varepsilon - z|^2 + Mv(x^\varepsilon) - Mv(y^\varepsilon).$$

Using $(\mu - 1)c(\xi^1) \le (\mu - 1)k$,

$$\Phi_\varepsilon(x^\varepsilon y^\varepsilon) \le w(x^\varepsilon + \xi^1) - v(x^\varepsilon + \xi^1) - C_\epsilon |x^\varepsilon + \xi^1 - z|^2$$

$$+\, C_\epsilon\left(|x^\varepsilon + \xi^1 - z|^2 - |y^\varepsilon - z|^2\right) + (\mu - 1)k$$

$$+\, Mv(x^\varepsilon) - Mv(y^\varepsilon).$$

Hence for ε small enough, using $(\mu - 1)k < 0$,

$$\Phi_\varepsilon(x^\varepsilon, y^\varepsilon) < \Phi_\varepsilon(x^\varepsilon + \xi^1, x^\varepsilon + \xi^1)$$

which contradicts our choice of $(x^\varepsilon, y^\varepsilon)$.

Notice this part is where C_ϵ is used.

CASE B: $v(y^\varepsilon) < Mv(y^\varepsilon)$.

Using (2.14) and (2.15),

$$\alpha(w(z) - v(z)) \leq \alpha(x^\epsilon)(w(x^\epsilon) - v(x^\epsilon)) + o_\epsilon(1) + \alpha(y^\epsilon)v(y^\epsilon) - \alpha(y^\epsilon)v(y^\epsilon)$$

$$\leq o_\epsilon(1) + \alpha(x^\epsilon)w(x^\epsilon) - \alpha(y^\epsilon)v(y^\epsilon)$$

$$\leq -2\left(g_\iota(y^\epsilon) - g_\iota(x^\epsilon)\right)\left(\frac{x^\epsilon - y^\epsilon}{\epsilon^2}\right) + f(y^\epsilon) - \mu f(x^\epsilon)$$

$$-2g_\iota(y^\epsilon)C_\epsilon(y^\epsilon - z)_i - \lambda(y^\epsilon)\int_\Omega (v(y) - v(y^\epsilon))\,Q(dy, y^\epsilon) \qquad (2.16)$$

$$+ \lambda(x^\epsilon)\int_\Omega (w(y) - w(x^\epsilon))\,Q(dy, x^\epsilon)$$

$$\leq C_f(1 - \mu) + o_\epsilon(1).$$

(Note that $o_\epsilon(1) \to 0$ as $\epsilon \to 0$.) See Lenhart [12] for an estimate on such integral terms; the key idea used is that

$$\int_\Omega [(w(y) - v(y)) - (w(z) - v(z))]\,Q(dy, x^\epsilon) \leq 0 \qquad (2.17)$$

and the integral terms in (2.16) are close to the integral terms in (2.17). Thus we conclude

$$\max_x (w(x) - v(x)) \leq C(1 - \mu) + o_\epsilon(1).$$

Let $\mu \to 1$,

$$\max_x (u(x) - v(x)) \leq o_\epsilon(1).$$

Then let $\epsilon \to 0$,

$$\max_x (u(x) - v(x)) \leq 0. \quad \blacksquare$$

Now we obtain a similar result in $\mathbf{R}^n$.

THEOREM 2.2. *Under assumptions (2.3)–(2.9) on $\mathbf{R}^n$, if u is a viscosity subsolution of (1.1) on $\mathbf{R}^n$ and v is a viscosity supersolution of (1.1) on $\mathbf{R}^n$, then*

$$u(x) \leq v(x) \text{ on } \mathbf{R}^n.$$

PROOF: Let $o < \delta < 1$. Using notation from proof of Theorem 2.1, choose z such that

$$w(z) - v(z) \geq \sup_x (w - v)(x) - \delta.$$

If $C_\epsilon |y - z|^2 + \left|\frac{x-y}{\epsilon}\right|^2 \geq 5\max(\|u\|_\infty, \|v\|_\infty)$, with $C_\epsilon = \sqrt{\omega_v\left(\sqrt{2M+1}\epsilon\right)}$,

$$\Phi_\epsilon(x, y) \leq \Phi_\epsilon(z, z)$$

$(x^{\epsilon\delta}, y^{\epsilon\delta})$. Thus Φ_ϵ does achieve its maximum at a finite point, say at $(x^\epsilon, y^\epsilon) = (x^{\epsilon\delta}, y^{\epsilon\delta})$. We obtain (2.12)–(2.15) as before, with $x^{\epsilon\delta} - y^{\epsilon\delta} \to 0$ as $\epsilon \to 0$ and $\delta \to 0$.

CASE A: $v(y^\epsilon) \geq Mv(y^\epsilon)$. By (2.8), there exists $\xi^1 \geq 0$ such that

$$v(y^\epsilon) > v(y^\epsilon + \xi^1) + c(\xi^1) - \frac{k(1-\mu)}{2}$$

and $|\xi^1| \leq C_1$ where C_1 depends on $\|v\|_\infty$ but not on ϵ, δ. Now

$$\Phi_\epsilon(x^\epsilon, y^\epsilon) \leq w(x^\epsilon) - v(y^\epsilon) - \left|\frac{x^\epsilon - y^\epsilon}{\epsilon}\right|^? - C_\epsilon |y^\epsilon - z|^2$$

$$\leq w(x^\epsilon + \xi^1) + \mu c(\xi^1) - v(y^\epsilon + \xi^1) - c(\xi^1) + \frac{k(1-\mu)}{2}$$

$$- \left|\frac{x^\epsilon - y^\epsilon}{\epsilon}\right|^2 - C_\epsilon |y^\epsilon - z|^2$$

$$\leq w(x^\epsilon + \xi^1) - v(y^\epsilon + \xi^1) - \left|\frac{x^\epsilon - y^\epsilon}{\epsilon}\right|^2 - C_\epsilon |y^\epsilon + \xi^1 - z|^2$$

$$+ C_\epsilon \left(|y^\epsilon + \xi^1 - z|^2 - |y^\epsilon - z|^2\right) + \frac{k}{2}(\mu - 1)$$

$$\leq \Phi_\epsilon(x^\epsilon + \xi^1, y^\epsilon + \xi^1) + o_{\epsilon\delta}(1) + \frac{k}{2}(\mu - 1)$$

$$< \Phi_\epsilon(x^\epsilon + \xi^1, y^\epsilon + \xi^1) \text{ for } \epsilon \text{ sufficiently small.}$$

This contradicts the choice of (x^ϵ, y^ϵ).

CASE B: $v(y^\epsilon) < Mv(y^\epsilon)$ follows as in Theorem 2.1. ∎

3. EXISTENCE RESULTS.

Due to the possible incompatibility of the impulse obstacle and boundary condition (1.2), we shall prove existence of viscosity solutions to

$$\max(Lu - f, u - Mu) = 0 \text{ in } \Omega \tag{3.1}$$

satisfying the boundary condition:

$$u(x) = Mu(x) \wedge \int_\Omega u(y)Q(dy, x) \text{ for } x\epsilon\partial\Omega \text{ ($\wedge$ minimum symbol).} \tag{3.2}$$

Condition (3.2) formally means the state process could jump back into the interior of Ω upon hitting $\partial\Omega$ or an impulse could be used to change the state. We have the following existence result.

THEOREM 3.1. *Under assumptions (2.3)–(2.10) on Ω, there is a unique viscosity solution of (3.1) on Ω satisfying (3.2).*

PROOF: For $\Psi\epsilon W^{1,\infty}(\overline{\Omega})$, by an extension of [5, 12], we have the existence of unique viscosity solution of

$$\max(Lu - f, u - \psi) = 0 \text{ on } \Omega \qquad (3.3)$$

satisfying

$$u(x) = \psi(x) \wedge \int_\Omega u(y)Q(dy, x) \text{ for } x \epsilon \partial\Omega. \qquad (3.4)$$

We also have the continuous dependence of u on ψ, i.e., if u_1, u_2 are viscosity solutions of (3.3) satisfying (3.4) with obstacles ψ_1, ψ_2, respectively,

$$\|u_1 - u_2\|_\infty \leq \|\psi_1 - \psi_2\|_\infty.$$

Using this continuous dependence result, we obtain the existence and uniqueness of viscosity solutions of (3.3) satisfying (3.4), for obstacles ψ in $C(\overline{\Omega})$. Now we will apply this result with $\psi = Mu$, with $u \epsilon C(\overline{\Omega})$ giving $Mu \epsilon C(\overline{\Omega})$ by (2.10).

We now construct a sequence which will converge to the solution of (3.1) satisfying (3.2). Define u_0 to be the unique viscosity solution of

$$Lu_0 = f \text{ in } \Omega$$
$$u_0(x) = \int_\Omega u_0(y)Q(dy, x) \text{ on } \partial\Omega.$$

By (2.10), $u_0 \epsilon C(\overline{\Omega})$ implies $Mu_0 \epsilon C(\overline{\Omega})$. Thus there exists a sequence of viscosity solutions, $\{u_n\}_{n=1}^\infty$, satisfying

$$\max(Lu_n - f, u_n - Mu_{n-1}) = 0 \text{ in } \Omega$$
$$u_n(x) = Mu_{n-1}(x) \wedge \int_\Omega u_n(y)Q(dy, x) \text{ on } \partial\Omega.$$

By a maximum principle argument on these viscosity solutions, we obtain

$$0 \leq u_n \leq u_{n-1}, \ n = 1, 2, \ldots \ .$$

To get uniform convergence of our sequence, first define a map

$$\sigma : C(\overline{\Omega}) \rightarrow C(\overline{\Omega}) \text{ by}$$

$$\sigma(\psi) = u \text{ where}$$

u is the viscosity solution of (3.3) satisfying (3.4).

The map $T = \sigma \circ M$ is increasing and concave. We have the following properties for T:

There exists $\overline{\lambda}$ in $(0,1)$ such that $\overline{\lambda}u_0 \leq k.$

If there exists $\beta \epsilon [0,1]$ such that

$$v - w \leq \beta w, \text{ then } Tv - Tw \leq (1-\lambda)\beta Tv. \tag{3.5}$$

See Hanouzet and Joly [13], Perthame [14, 15] and Barles [8, 16] for examples of this technique. Since $u_0 - u_1 \geq u_0$, we iterate (3.5) to obtain

$$u_{n-1} - u_n \leq (1 - \overline{\lambda})^{n-1} u_0.$$

In this way, we obtain uniform convergence, $u_n \to u$. The uniform convergence of u_n insures the convergence of the integral terms and Mu_{n-1}, and we have that u satisfies boundary condition (3.2).

To show the limit function u is a viscosity solution, we first look at point x_0 where $u - \phi$ has a global minimum. Then there is a sequence $\{x_n\}$ converging to x_0 such that $u_n - \phi$ has its minimum at x_n. Since $u_n \leq Mu_{n-1}$ on Ω, we know

$$u \leq Mu \text{ on } \Omega.$$

If $u = Mu$ at x_0, then the viscosity supersolution condition (2.2) is satisfied. If $u < Mu$ at x_0, then $u_n(x_n) < Mu_{n-1}(x_n)$ for n large enough, which implies

$$-g_i\phi_{x_i} + \alpha u_n - \lambda \int_{\Omega} (u_n(y) - u_n(x_n))\, Q(dy, x_n) \geq f \text{ at } x_n.$$

Letting $n \to \infty$, we have u is a viscosity supersolution. The subsolution case follows similarly.

The uniqueness result follows as in Theorem 2.1 except for the case when the maximum of $w - v$ occurs at $z \epsilon \partial \Omega$. If

$$v(z) \geq \int_{\Omega} v(y) Q(dy, x)$$

then the argument goes as before. If $v(z) \geq Mv(z)$, then there exists nonzero $\xi^1 \geq 0$ such that

$$Mv(z) = v(z + \xi^1) + c(\xi^1).$$

Then

$$w(z) - v(z) \leq w(z + \xi^1) + \mu c(\xi^1) - v(z + \xi^1) - c(\xi^1)$$

$$\leq w(z + \xi^1) - v(z + \xi^1) + (\mu - 1)k$$

$$< w(z + \xi^1) - v(z + \xi^1),$$

which is a contradiction of the choice of z. ∎

The existence proof for the $\mathbf{R}^n$ case is simpler.

THEOREM 3.2. *Under assumptions (2.3)–(2.9) on $\mathbf{R}^n$, there exists a unique viscosity solution to (1.1) in $\mathbf{R}^n$.*

PROOF: We use the iterative approximation scheme on $\mathbf{R}^n$,

$$\max(Lu_n - f, u_n - Mu_{n-1}) = 0 \text{ in } \mathbf{R}^n, \ n = 1, 2, \ldots \text{ and } Lu_0 = f \text{ in } \mathbf{R}^n. \qquad (3.6)$$

To obtain existence for (3.6), we use extensions of results from [5, 12] with ψ in $BUC(\mathbf{R}^n)$. Note that $u_{n-1} \epsilon BUC(\mathbf{R}^n)$ implies $Mu_{n-1} \epsilon BUC(\mathbf{R}^n)$. By using an operator T as in Theorem 3.1, we obtain uniform convergence of $\{u^n\}$ to a viscosity solution u. The uniqueness result is Theorem 2.2. ∎

4. CONTROL REPRESENTATION.

To put the results obtained in section 3 in the context of the classical results on impulse control (Bensoussan and Lions [17] and Menaldi [18]) we show that the solution obtained in Theorem 3.2 is equal to the value function associated with the impulse control problem (1.4).

THEOREM 4.1. *The unique viscosity solution $u(x)$ from Theorem 3.2 is equal to the value function $V(x)$ from (1.4)*

PROOF: First, we show that the approximations u_n have the following control interpretation:

$$u_n(x) = \inf\{J_x(v_n) : v_n \text{ impulse control strategy with } \theta_i = \infty, \text{ for all } i \geq n+1\}, \qquad (4.1)$$

i.e., u_n is the minimum cost function associated with the impulse control problem with at most n impulses allowed. We show (4.1) by induction. Call the right hand side of (4.1), $V_n(x)$. The representation for u_0 (no impulses) is valid. Assume (4.1) for u_{n-1}. By [5,12], we have

$$u_n(x) = \inf_{\theta} E_x \left[\int_0^\theta f(x(t))e^{-\alpha t}dt + Mu_{n-1}(x(\theta))e^{-\alpha\theta} \right] \qquad (4.2)$$

where θ is a stopping time. Let v_n be an impulse control strategy with n impulses,

$$v_n = (\theta_1, \xi_1, \ldots, \theta_n, \xi_n).$$

By (4.2), and writing $x_n(t) = x^{v_n}(t)$ from (1.3),

$$u_n(x) \le E_x \left[\int_0^{\theta_1} f(x_n(t))e^{-\alpha t}dt + Mu_{n-1}(x_n(\theta_1 - 0))e^{-\alpha\theta_1} \right]$$

$$\le E_x \left[\int_0^{\theta_1} f(x_n(t))e^{-\alpha t}dt + e^{-\alpha\theta_1}(u_{n-1}(x_n(\theta_1 - 0) + \xi_1) + c(\xi_1)) \right]$$

$$\le E_x \left[\int_0^{\theta_1} f(x_n(t))e^{-\alpha t}dt + e^{-\alpha\theta_1}c(\xi_1) \qquad (\text{use } (4.1) \text{ on } u_{n-1}) \right.$$

$$\left. + e^{-\alpha\theta_1} E_{x_n(\theta_1)+\xi_1} \left(\int_{\theta_1}^{\infty} f(x_n(t))e^{-\alpha(t-\theta_1)}dt + \sum_{i=2}^{n} e^{-\alpha(\theta_i - \theta_1)}c(\xi_i) \right) \right]$$

Therefore

$$u_n(x) \le V_n(x).$$

To show $u_n \ge V_n$, for $\varepsilon > 0$, choose θ_1 such that

$$E_x \int_0^{\theta_1} f(x(t))e^{-\alpha t}dt + e^{-\alpha\theta_1} Mu_{n-1}(x(\theta_1 - 0)) < u_n(x) + \varepsilon/2 \qquad (4.3)$$

Using (2.8), choose ξ_1 such that

$$Mu_{n-1}(x(\theta_1 - 0)) = u_{n-1}(x(\theta_1 - 0) + \xi_1) + c(\xi_1). \qquad (4.4)$$

By the inductive hypothesis on $u_{n-1}(x(\theta_1 - 0) + \xi_1)$, there exists an impulse control strategy

$$v_{n-1} = (\theta_2, \xi_2, \dots, \theta_n, \xi_n), \qquad \theta_2 > \theta_1,$$

such that

$$J_{x(\theta_1)+\xi_1}(v_{n-1}) < u_{n-1}(x(\theta_1 - 0) + \xi_1) + \varepsilon/2 \qquad (4.5)$$

By (4.3)-(4.5),

$$E_x \left[\int_0^{\theta_1} f(x(t))e^{-\alpha t}dt + e^{-\alpha\theta_1} E_{x(\theta_1)+\xi_1} \left[\int_{\theta_1}^{\infty} f(x_{n-1}(t))e^{-\alpha(t-\theta_1)}dt \right. \right.$$

$$\left. \left. + \sum_{i=1}^{n} e^{-\alpha(\theta_i - \theta_1)}c(\xi_i) - \varepsilon/2 \right] \right] < u_n(x) + \varepsilon/2$$

which implies

$$V_n(x) \le J_x(v_n) < u_n(x) + \varepsilon,$$

where $v_n = (\theta_1, \xi_1, v_{n-1})$. This completes the proof of (4.1).

By construction, $u_n \searrow u$. Thus, by (4.1),

$$u(x) \le \inf_{v_n} J_x(v_n) \text{ for all } n.$$

For $\varepsilon > 0$, there exists a strategy v_m such that

$$u(x) \leq J_x(v_m) < V(x) + \epsilon.$$

Thus $u \leq V$. To show $u \geq V$, there exists index j large enough and strategy v_j, such that

$$J_x(v_j) < u_j(x) + \epsilon/2 < u(x) + \epsilon$$

and hence $V(x) \leq u(x)$. ∎

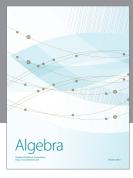Similar control representations can be obtained in the bounded domain case.
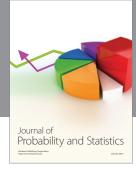
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