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QUASIASYMPTOTIC EXPANSION OF DISTRIBUTIONS FROM S'+ AND THE ASYMPTOTIC EXPANSION OF THE DISTRIBUTIONAL STIELTJES TRANSFORM

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ABSTRACT. By following the approach of Drožinov, Vladimirov and Zavialov we investigate the quasiasymptotic expansion of distributions and give Abelian type results for the ordinar asymptotic behaviour of the distributional Stieltjes transform of a distribution with appropriate quasiasymptotic expansion.

KEY WORDS AND PHRASES. Stieltjes transform of distributions, quasiasymptotic behaviour of distributions.
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1. INTRODUCTION.

In the papers [9,10] authors followed the definition of the distributional Stieltjes transform given in [7] which enabled them to use the strong theory of the space of tempered distributions S'. In fact, they generalized slightly the definition of Lavoine and Misra. Using the notion of the quasiasymptotic behaviour of distributions from $S'_+ = \{f \in S', \text{supp } f \subset [0,\infty)\}$, introduced by Zavialov in [15], they obtained more general results than in [6,7,2] for the asymptotic behaviour of the distributional Stieltjes transform at ∞ and 0^+. Let us notice that the notion of the quasiasymptotic behaviour of distributions was studied by Drožinov, Vladimiraov and Zavialov in several papers (see [13] and references there) in which they obtained remarkable results in the quantum fiela theory.

McClure and Wong [3,14] studied the asymptotic expansion of the generalized Stieltjes transform of some classes of locally integrable functions characterized by their asymptotic expansions at ∞ and 0^+.

Our approach to the asymptotic expansion of the distributional Stieltjes transform which we study in this paper is quite different from the approach given in [3,14].

In the first part of the paper we slightly extend and investigate the definition of the quasiasymptotic expansion at ∞ of a distribution from S'_+ given in [4, p. 385]. Also, we give the definition of the quasiasymptotic expansion at 0^+ of an element from S'_+.

This enables us to obtain, in the second part of the paper, the asymptotic expansions at ∞ and at 0^+ of the Stieltjes transforms of appropriate distributions from S'_+.

Domains in [14] and in this paper on which the Stieltjes transform is defined do not contain each other. The examples given at the end of this paper show some advantages of our approach in the case when a distribution belongs to the intersection of the mentioned domains.

2. NOTATION

As usually, R, C, N, are the spaces of real, complex and natural numbers; $N_0 = N \cup \{0\}$.

The space of rapidly decreasing functions is denoted by S and by S_m, $m \in N_0$, the completion of S under the norm

$$\|\phi\|_m = \sup\{ (1 + x^2)^{m/2} |\phi^{(i)}(x)|;\ x \in R,\ i \leq m\}.$$

A positive continuous function L defined on $(0,\infty)$ is called slowly varying at ∞ (0^+) if for every $a > 0$

$$\lim_{t\to\infty} L(at)/L(t) = 1 \quad (\lim_{t\to 0^+} L(at)/L(t) = 1).$$

We denote by Σ_∞ (Σ_{0^+}) the set of all slowly varying (in short s.v.) functions at ∞ (0^+). For the properties of s.v. functions we refer the reader to [11].

If L is an s.v. function at ∞ (0^+), then ([11]) for every $\varepsilon > 0$ there is $A_\varepsilon > 0$ so that

$$x^{-\varepsilon} < L(x) < x^{\varepsilon} \ (x^{-\varepsilon} > L(x) > x^{\varepsilon}) \quad \text{if} \quad x > A_\varepsilon \quad (0 < x < A_\varepsilon).$$

This property of L and the corresponding properties of S_m ([12, p. 93]) imply the following assertion which we shall use in parts 6 and 7:

$$\left.\begin{array}{l} \text{Let } G \in L^1_{loc},\ \text{supp } G \subset [0,\infty),\ \alpha > -1 \text{ and } G(x) \sim x^\alpha L(x) \text{ as } x \to \infty\ (x \to 0^+). \\ \text{Then } G(kx)/(k^\alpha L(k)) \to x^\alpha_+,\ k \to \infty, \text{ in } S'_t \text{ for } t > \alpha+1. \\ (\text{If } t > \alpha+1 \text{ and } G \in S'_t \text{ then } G(x/k)/((1/k)^\alpha L(1/k)) \to x^\alpha_+,\ k \to \infty, \text{ in } S'_t.) \end{array}\right\} \quad (2.1)$$

Recall, for $\alpha > -1$, $x^\alpha_+ = H(x)x^\alpha$; H is Heviside's function. (The symbol $\sim$ is related to the ordinary asymptotic behaviour.)

The following scale of distributions from S' has been used in investigations of the quasiasymptotic behaviour of distributions:

$$f_{\alpha+1} = \begin{cases} Ht^\alpha/\Gamma(\alpha+1), & \alpha > -1 \\ D^n f_{\alpha+n+1}, & \alpha \leq -1,\ \alpha+n > -1 \end{cases} \qquad ([12, \text{p. } 88])$$

where D is the distributional derivative.

3. THE q.a.b. OF DISTRIBUTIONS

We shall repeat in this section some well-known facts about the quasiasymptotic behaviour from [13,10].

Let $f \in S'_+$. It is said that f has the quasiasymptotic behaviour (in short q.a.b.) at ∞ (0^+) with the limit $g \neq 0$ with respect $k^\alpha L(k)$, $L \in \Sigma_\infty$ $((1/k)^\alpha L(1/k),\ L \in \Sigma_0)$, $\alpha \in R$, if

$$\left.\begin{array}{l}\lim\limits_{k\to\infty} < f(kt)/(k^{\alpha}L(k)),\phi(t) > = < g(t),\phi(t) >,\ \phi \in S \\ (\lim\limits_{k\to\infty} < f(t/k)/((1/k))^{\alpha}L(1/k)),\phi(t) > = < g(t),\phi(t) >,\ \ \phi \in S).\end{array}\right\} \quad (3.1)$$

Let us notice that in [10] we reformulate the definition of the q.a.b. at 0^+ from [13].

We need in the paper the following structural theorem (for the q.a.b. at ∞ see [13] and for the q.a.b. at 0^+ see [10]):

$$\left.\begin{array}{l}\text{Let } f \overset{q.a.}{\sim} g \text{ at } \infty\ (0^+) \text{ with respect to } k^{\alpha}L(k)\ ((1/k)^{\alpha}L(1/k)). \\ \text{Then there exist } F \in L^1_{loc},\ \text{supp } F \subset [0,\infty),\ C \neq 0 \text{ and } m \in N_0,\ m+\alpha > -1, \\ \text{such that } f = D^m F,\ F(k) \sim Ck^{m+\alpha}L(k),\ k \to \infty \quad (F(1/k) \sim C(1/k)^{m+\alpha}L(1/k), \\ k \to \infty).\end{array}\right\} \quad (3.2)$$

EXAMPLES

1. For $\alpha < -1$, $H(x-1)x^{\alpha} \overset{q.a.}{\sim} \frac{1}{-\alpha+1}\delta(x)$ at ∞ with respect to k^{-1};

Moreover, if $f \in L^1(R)$, supp $f \subset [0,\infty)$, $\int f = C \neq 0$, then $f \sim C\delta$ at ∞ with respect to k^{-1};

2. $H(x-1)/x \overset{q.a.}{\sim} \delta$ at ∞ with respect to $k^{-1}\ln k$;

3. Any distribution with compact support has the q.a.b. at ∞ with respect to k^{-m} for some $m \in N$;

4. $H(x-1)e^{1/x} \overset{q.a.}{\sim} H(x)$ at ∞ with respect to $k^0 = 1$;

5. $x_+^{-m} \overset{q.a.}{\sim} \frac{(-1)^{m-1}}{(m-1)!}\delta^{(m-1)}$ at ∞ with respect to $k^{-m}\ln k$, $(m \in N)$;

6. For $\lambda > -1$, $m \in N_0$, $(x^{\lambda}\ln^m x)_+ \overset{q.a.}{\sim} x_+^{\lambda}$ at 0^+ with respect to $(1/k)^{\lambda}\ln^m(1/k)$;

7. For $-n > \lambda > -n-1$, $(x^{\lambda}\ln^m x)_+ \overset{q.a.}{\sim} \Gamma(\lambda+1)D^n f_{\lambda+n+1}$ at 0^+ with respect $(1/k)^{\lambda}\ln^m(1/k)$;

8. For $n \in N$, $m \in N_0$, $(x^{-n}\ln^m x)_+ \overset{q.a.}{\sim} \frac{(-1)^{n-1}}{(m+1)(n-1)!}\delta^{(n-1)}$ at 0^+ with respect to $(1/k)^{-n}\ln^{m+1}(1/k)$.

For the definition of distributions x_+^{-m}, $m \in N$, $(x^{\lambda}\ln^m x)_+$, $-n > \lambda > -n-1$, $(x^{-n}\ln^m x)_+$, $n \in N_0$, see [5, pp. 338, 339].

We remark that the q.a.b. at 0^+ is a local property while the q.a.b. at ∞ is a global property of an $f \in S'_+$.

Namely, it is proved in [13] that if $f = 0$ in a neighbourhood of zero then for any α and L

$$\lim_{k\to\infty} f(x/k)((1/k)^{\alpha}L(1/k)) = 0 \ \text{ in } \ S'.$$

Clearly this does not hold for ∞ (see example 3.).

As it was shown in [9,10] the notion of q.a.b. is much more appropriate for the investigations of the Stieltjes transform. For example if $f \in L^1$, supp $f \subset [0,\infty)$ and $f \sim 1/x^5$ as $x \to \infty$, this ordinar asymptotic does not imply the behaviour of its Stieltjes transform. The behaviour of the transform is determined by the quasiasymptotic behaviour of f (see example 1.).

4. THE q.a.e. OF DISTRIBUTIONS

We extend slightly the definitions of the closed and open quasiasymptotic expansion, in short the q.a.e. at ∞, given in [4] and using the same idea we give the definition of the q.a.e. at 0^+.

Let $\alpha \in R$ and $L \in \Sigma_\infty$ ($L \in \Sigma_0$). We put

$$(f_L)_{\alpha+1} = \begin{cases} H(t)L(t)t^\alpha/\Gamma(\alpha+1), & \alpha > -1, \\ D^n(f_L)_{\alpha+n+1}, & \alpha \leq -1,\ \alpha+n > -1, \end{cases} \tag{4.1}$$

where n is the smallest natural number such that $\alpha+n > -1$.

Obviously, $(f_L)_{\alpha+1} \overset{q.a.}{\sim} f_{\alpha+1}$ at ∞ (0^+) with respect to $k^\alpha L(k)$ ($(1/k)^\alpha L(1/k)$).

DEFINITION 1. We say that an $f \in S'_+$ has the closed q.a.e. at ∞ (0^+) of order $(\alpha,L) \in R \times \Sigma_\infty$ ($(\alpha,L) \in R \times \Sigma_0$) and of lenght ℓ, $0 \leq \ell < \infty$, with respect to $k^{\alpha-\ell}L_0(k)$ ($(1/k)^{\alpha+\ell} \cdot L_0(1/k)$) if f has the q.a.b. at ∞ (0^+) with respect to $k^\alpha L(k)$ ($(1/k)^\alpha L(1/k)$) and if there exist $\alpha_i \in R$, $L_i \in \Sigma_\infty$ ($L_i \in \Sigma_{0^+}$), $c_i \in C$, $i = 1,\dots,N$, $N \in N$, $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_N$ ($\alpha_1 \leq \alpha_2 \dots \leq \alpha_N$) and that f is of the form

$$f(t) = \sum_{i=1}^{N} c_i (f_{L_i})_{\alpha_i+1}(t) + h(t) \tag{4.2}$$

such that

$$\lim_{x\to\infty} \left< \frac{h(kt)}{k^{\alpha-\ell}L_0(k)}, \phi(t) \right> = 0,\ \phi \in S \tag{4.3}$$
$$\left(\lim_{k\to\infty} \left< \frac{h(t/k)}{(1/k)^{\alpha+\ell}L_0(1/k)}, \phi(t) \right> = 0,\ \phi \in S\right).$$

Obviously, we shall assume that $c_i \neq 0$ and that $\alpha_N \geq \alpha-\ell$ ($\alpha_N \leq \alpha+\ell$).

Since the sum of two slowly varying functions is the slowly varying one we can and we shall always assume that in the representation (4.2) $\alpha_1 > \alpha_2 > \dots > \alpha_N$ ($\alpha_1 < \alpha_2 < \dots \alpha_N$). Namely, $(f_{L_j})_{\beta+1} + (F_{L_k})_{\beta+1} = (f_{L_j+L_k})_{\beta+1}$.

$(f_{L_j})_{\beta_1+1}$ and $(f_{L_k})_{\beta_1+1}$ have the same q.a.b. at ∞ (0^+) iff $\beta_1 = \beta_2$ and $L_j \sim L_k$. So, we have:

PROPOSITION 1. Let $f \in S'_+$ satisfy conditions of Definition 1 and assume that there are two representations of f

$$f(t) = \sum_{i=1}^{N} c_i (f_{L_i})_{\alpha_i+1} + h(t),$$
$$f(t) = \sum_{j=1}^{M} \tilde{c}_i (f_{L_i})_{\tilde{\alpha}_i+1} + \tilde{h}(t)$$

for which all the assumptions given above hold. Then $M = N$, $\alpha_1 = \tilde{\alpha}_1, \dots, \tilde{\alpha}_N = \tilde{\alpha}_N$, $L_1 \sim \tilde{L}_1, \dots, L_N \sim \tilde{L}_N$, $\alpha_1 = \alpha$, $L_1 \sim L$.

We shall use the following notation for the $f \in S'_+$ from Definition 1:

$$f \overset{q.e.}{\sim} \sum_{i=1}^{N} c_i (f_{L_i})_{\alpha_i+1} \text{ at } \infty\ (0^+) \text{ of order } (\alpha,L) \tag{4.4}$$

with respect to $k^{\alpha-\ell}L_0(k)$ ($(1/k)^{\alpha+\ell}L_0(1/k)$).

EXAMPLES

9. We haw that $\sum_{r=1}^{\infty} \frac{H(x-1)}{r!x^r}$ uniformly converges to $H(x-1)e^{1/x}$ but

$$H(x-1)e^{1/x} \overset{q.e.}{\sim} H(x) + ((\ln x)_+)' \text{ at } \infty \text{ of order } (0, L \equiv 1) \tag{4.5}$$

with respect to $k^{-1}\ln k$ and

$$H(x-1)e^{1/x} \overset{q.e.}{\sim} H(x) + ((\ln x)_+)' + \left(-1+\sum_{r=2}^{\infty} \frac{1}{r!}\,\frac{1}{r-1}\right)\delta(x) \tag{4.6}$$

of the order $(0,L\equiv 1)$ with respect to k^{-1}.

10. $H(t-1)/t \overset{q.e.}{\underset{\infty}{}} \delta + \delta'$ at ∞ of order $(-1,L\equiv 1)$ with respect to $k^{-2}\ln k$. Moreover, let $n > 2$; then for $j \leq n-1$

$$H(t-1)/t^n \overset{q.e.}{\sim} \frac{1}{(n-1)}\delta + \frac{1}{(n-2)1!}\delta' + \dots + \frac{(-1)^{j-1}\delta^{(j-1)}}{(n-1)(j-1)!} \tag{4.7}$$

at ∞ of order $(-1.L\equiv 1)$ with respect to k^{-j};

$$H(t-1)/t^n \overset{q.e.}{\sim} \frac{1}{(n-1)}\delta + \frac{1}{(n-2)1!}\delta' + \dots + \frac{(-1)^{n-2}}{(n-2)!}\delta^{(n-2)} + \frac{(-1)^{n-1}\delta^{(n-1)}}{(n-1)!} \tag{4.8}$$

at ∞ of order $(-1,L\equiv 1)$ with respect to $k^{-n}\ln k$.

11.

$$H(1-x)x_+^{-m} \overset{q.e.}{\sim} \frac{(-1)^{m-1}}{(m-1)!}(\ln x)_+^{(m)} + \frac{(-1)^{m-1}}{(m-1)!}\left(\sum_{i=1}^{m-1}\frac{1}{i}\right)\delta^{(m-1)}(x) \tag{4.9}$$

at 0^+ of order $(-m,\ln(1/k))$ with respect to $(1/k)^{-m}$.

Following [4] we define the open q.a.e.

DEFINITION 2. An f has the open q.a.e. at ∞ (0^+) of order $(\alpha,L) \in R\times\Sigma)$ $((\alpha,L)\in R\times \Sigma_0)$ and of length s, $0 < s \leq \infty$, iff for every ℓ, $0 \leq \ell < s$, f has the closed q.a.e. of order (α,L) and of length ℓ, with respect to $k^{\alpha-\ell}L_\ell(k)$ $((1/k)^{\alpha+\ell}L_\ell(1/k))$.

By the same arguments as for Proposition 1 one can prove the following proposition:

PROPOSITION 2. Let f have the open q.a.e. at ∞ of order (α,L) and of length s and let $0 \leq \ell_1 < \ell_2 < s$. Suppose that

$$f \overset{q.e.}{\sim} \sum_{i=1}^{N} a_i(f_{L_i})_{\alpha_i+1} \text{ at } \infty\ (0^+) \text{ with respect to } k^{\alpha-\ell_1}L_{\ell_1}(k)\ \ ((1/k)^{\alpha+\ell_1}L_{\ell_1}(1/k)),$$

$$f \overset{q.e.}{\sim} \sum_{i=1}^{M} b_i(f_{L_i})_{\beta_i+1} \text{ at } \infty\ (0^+) \text{ with respect to } k^{\alpha-\ell_2}L_{\ell_2}(k)\ \ ((1/k)^{\alpha+\ell_2}L_{\ell_2}(1/k)).$$

Then, $M \geq N$ and $a_i = b_i$, $\alpha_i = \beta_i$, $L_i \sim \tilde{L}_i$, $i = 1,\dots,N$.

Let us note if f has the closed q.a.e. at ∞ of order (α,L) with respect to $k^{\alpha-\ell}L(k)$ then for any $s \leq \ell$, f has the open q.a.e. at ∞ of order (α,L) and of length s. The similar conclusion holds for the point 0^+, as well.

Proposition 2 implies:

CORROLARY 3. Let f have the open q.a.e. at ∞ (0^+) of order (α,L) and of lenght s. Then f may be asymptotically expanded into a series

$$f(x) \overset{q.e.}{\sim} \sum_{i=1}^{\infty} c_i(f_{L_i})_{\alpha_i+1} \text{ at } \infty\ (0^+)$$

where $\alpha = \alpha_1 > \dots > \alpha_n > \dots$, $(\alpha = \alpha_1 < \dots < \alpha_n < \dots)$, so that for any $0 \leq \ell < s$ and L

$$(f - \sum_{i=1}^{\infty} c_i(f_{L_i})_{\alpha_i+1})(kx)/(k^{\alpha-\ell}L(k)) \to 0 \text{ in } S'), \quad k \to \infty$$

$$((f - \sum_{i=1}^{\infty} c_i(f_{L_i})_{\alpha_i+1})(x/k)/((1/k)^{\alpha+\ell}L(1/k)) \to 0 \text{ in } S', \ k \to \infty).$$

(Note that here c_i can be equal to zero for $i \geq N \in \mathbb{N}$).

5. THE DISTRIBUTIONAL STIELTJES TRANSFORM

There are several definitions of the Stieltjes transform of generalized functions. We follow the definition given by Lavoine and Misra [7]. Some advantages of this definition were mentioned in [8].

The space $J'(r)$, $r \in \mathbb{R} \setminus (-\mathbb{N})$, is a subspace of S'_+ such that $f \in J'(r)$ if there exist $m \in \mathbb{N}$ and $F \in L^1_{loc}$, supp $F \subset [0,\infty)$, such that

$$f = D^m F, \tag{5.1}$$

$$\int_0^{\infty} |F(t)|(t+\beta)^{-r-m-1}dt < \infty \text{ for } \beta > 0. \tag{5.2}$$

The Stieltjes transform S_r, $r \in \mathbb{R} \setminus (-\mathbb{N})$, of an $f \in J'(r)$ with the properties given in (5.1) and (5.2) is a complex valued function $S_r f$ defined by

$$(S_r f)(z) = (r+1)_m \int_0^{\infty} F(t)(t+z)^{-r-m-1}dt, \quad z \in \mathbb{C} \setminus (-\infty,0]. \tag{5.3}$$

(If $p \in \mathbb{R}$, $t \in \mathbb{N}$, $(p)_t = p(p+1) \ldots (p+t-1)$, $(p)_0 = 1$.)
It is proved in [7] that $S_r f$ is a holomorphic function in $\mathbb{C} \setminus (-\infty,0]$. If $f \in J'(r+m)$, then $D^m f \in J'(r)$ and

$$S_r(D^m f) = (r+1)_m(S_{r+m}f). \tag{5.4}$$

One can show easily that

$$D^m(S_r f) = (-1)^m(r+1)_m(S_{r+m}f), \quad f \in J'(r), \ m \in \mathbb{N}. \tag{5.5}$$

6. ON THE BEHAVIOUR OF $S_r f$

Let f have the q.a.b. at ∞ (0^+) with respect to $k^{\alpha}L(k)$ ($(1/k)^{\alpha}L(1/k)$). Then for some $m \in \mathbb{N}_0$, $m+\alpha > -1$, and $F \in L^1_{loc}$, supp $F \subset [0,\infty)$, (3.2) holds. In the case of the q.a.b. at ∞ this implies that $f \in J'(r)$ for $r > \alpha$, $r \in \mathbb{R} \setminus (-\mathbb{N})$. In the case of the q.a.b. at 0^+, f ought not belong to $J'(r)$ for $r > \alpha$, $r \in \mathbb{R} \setminus (-\mathbb{N})$. If $f \in J'(r)$ then $F \in S'_{r+m+1}$; this follows from [12, p. 93]. So, for $p \geq r+m+1 > \alpha+m+1$ (2.1) and (2.2) imply:

$$\left.\begin{array}{l} F(kt)/(k^{\alpha+m}L(k)) \to Cf_{\alpha+m+1} \text{ in } S'_p \quad k \to \infty \\ (\text{Let } f \in J'(r); \text{ then } F(t/k)/((1/k)^{\alpha+m}L(1/k)) \to Cf_{\alpha+m+1} \text{ in } S'_p \quad k \to \infty. \end{array}\right\} \tag{6.1}$$

For a given $z \in \mathbb{C} \setminus (-\infty,0]$ we denote by $A(z)$ the set of all $\eta(t) \in C^{\infty}$ such that $\eta \in A(z)$ if there is an $\varepsilon = \varepsilon_{\eta}$, $0 < 2\varepsilon < |\mathrm{Re}\, z|$, such that

$$0 \leq \eta(t) \leq 1, \ \eta(t) = 1 \text{ for } t > -\varepsilon, \ \eta(t) = 0 \text{ for } t < -2\varepsilon.$$

Clearly, for a given $z \in \mathbb{C} \setminus (-\infty,0]$ and every $\eta \in A(z)$

$$R \ni t \to \eta(t)(t+z)^{-r-m-1} \in S_p \text{ for } p < r+m+1. \tag{6.2}$$

For the main results of this section we need the following assertion from [10] ([9]):

$$\left.\begin{array}{l}\text{Let } f \in J'(r). \text{ We have } (x > 0,\ t > 0),\ (S_r f(tx) = x(r+1)\int_t^\infty (S_{r+1}f)(xu)du, \\ \text{and if } (S_{r+1}f)(x) \sim x^{-(r-\alpha)-1}L(x) \text{ as } x \to \infty\ \ (x \to 0^+) \text{ with } r > \alpha, \text{ then} \\ (S_r f)(x) \sim ((r+1)/(r-\alpha))x^{-(r-\alpha)}L(x) \text{ as } x \to \infty\ (x \to 0^+).\end{array}\right\} \tag{6.3}$$

Now, we are ready to prove:

THEOREM 4. Let f have the closed q.a.e. at ∞ of order (α,L) and of length ℓ with respect to $k^{\alpha-\ell}L_0(k)$ (see the notation in Definition 1).

Let $r > \alpha$, $r \in R \setminus (-N)$. Then

(i) $f \in J'(r)$, $(f_{L_i})_{\alpha_i+1} \in J'(r)$, $i = 1,\dots,N$;

(ii) If we put $S_r(f_{L_i})_{\alpha_i+1}(x) = s_{\alpha_i,L_i}(x)$, $i = 1,\dots,N$, then for $\tilde{L}_i \sim L_i$

$$s_{\alpha_i,L_i}(x) \underset{N}{\sim} s_{\alpha_i,\tilde{L}_i}(x) \sim \frac{\Gamma(r-\alpha_i)}{\Gamma(r+1)} x^{\alpha_i-r}L_i(x),\ x \to \infty.$$

(iii) $$(S_r f)(x) - \sum_{i=1}^{\ell} c_i \frac{\Gamma(r-\alpha_i)}{\Gamma(r+1)} x^{\alpha_i-r}L_i(x) = o(x^{\alpha-\ell}L_0(x)),\ x \to \infty. \tag{6.4}$$

PROOF. We shall prove the theorem by using the similar idea as in the proof of the main theorem in [9].

Obviously, (i) follows from (3.2).

(ii) Let $\beta < r-1$, $x \in R$, $L \in \Sigma_\infty$. Let m be the smallest element from N_0 such that $\beta+m > -1$. Then

$$S_r(f_L)_{\beta+1}(x) = (r+1)_m \int_0^\infty \frac{f_{\beta+m+1}(t)L(t)}{(x+t)^{r+m+1}}\,dt = (r+1)_m < f_{\beta+m+1}(t)L(t), \frac{\eta(t)}{(x+t)^{r+m+1}} >,$$

$\eta \in A(x)$,

where

$$< f_{\beta+m+1}(t)L(t), \frac{\eta(t)}{(x+t)^{r+m+1}} >$$

is observed as a pair from (S'_{r+m}, S_{r+m}). Obviously this pair does not depend on $\eta \in A(x)$. Since $r+m > \beta+m+1$, we have

$$S_r(f_L)_{\beta+1}(kx)/k^{\beta-r}L(k) = (r+1)_m < f_{\beta+m+1}(t)L(t), \frac{\eta(t)}{k^{\beta+m+1}L(k)(x+t/k)^{r+m+1}} >$$
$$= (r+1)_m < \frac{f_{\beta+m+1}(kt)L(kt)}{k^{\beta+m}L(k)}, \frac{\eta(kt)}{(x+t)^{r+m+1}} >$$
$$= (r+1)_m < \frac{f_{\beta+m+1}(kt)L(kt)}{k^{\beta+m}L(k)}, \frac{\eta(t)}{(x+t)^{r+m+1}} >.$$

If $k \to \infty$, from (6.1) it follows

$$S_r(f_L)_{\beta+1}(kx)/k^{\beta-r}L(k) \to < f_{\beta+m+1}(t), \frac{\eta(t)}{(x+t)^{r+m+1}} >$$
$$= \frac{(r+1)_m}{\Gamma(\beta+m+1)}\int_0^\infty \frac{t^{\beta+m}dt}{(x+t)^{r+m+1}} = \frac{\Gamma(r-\beta)}{\Gamma(r+1)}x^{\beta-r}.$$

On putting $x = 1$ we obtain that (ii) holds for all $\alpha_i < r-1$. Let us suppose that $r-1 \leq \beta < r$.

Then, by the same arguments given above, we have

$$(S_{r+1}(f_L)_{\beta+1})(x) \sim \frac{\Gamma(r+1-\beta)}{\Gamma(r+2)} x^{\beta-r-1}L(x),\ x \to \infty.$$

Now by (6.3) we complete the proof of (ii).

(iii) We can assume that $\alpha < r-1$ because if $r-1 \leq \alpha < r$ we have, as in (ii), to observe firstly $S_{r+1}(f_L)_{\beta+1}$ and after that to use (6.3). Since

$$f - \sum c_i(f_{L_i})_{\alpha_i+1} \in S'_{r+m},$$

(6.2) implies that in the sense of the dual pair (S'_{r+m}, S_{r+m}) we have

$$\{(S_r f)(kx) - \sum_{i=1}^{N} c_i\, S_r(f_{L_i})_{\alpha_i+1})(kx)\}/(k^{\alpha-\ell-r}L_0(x)) =$$

$$= \langle \{f(kt) - \sum_{i=1}^{N} c_i(f_{L_i})(kt)\}/(k^{\alpha-\ell}L_0(k)), \eta(t)(x+t)^{-r-m-1}\rangle \to 0 \text{ as } k \to \infty.$$

On putting $x = 1$ the assertion (iii) follows.

The similar assertion holds for the closed q.a.e. at 0^+ but with more restrictive assumptions.

THEOREM 5. Let f have the closed q.a.e. at 0^+ of order (α,L) and of length ℓ with respect to $(1/k)^{\alpha+\ell}L_0(1/k)$. If $\alpha+\ell < r$ and $f \in J'(r)$ then

$$(S_r f)(x) - \sum c_i \frac{\Gamma(r-\alpha_i)}{\Gamma(r+1)} x^{\alpha_i-r}L_i(x) = o(x^{\alpha+\ell-r}L_0(x)),\ x \to 0. \tag{6.5}$$

The proof of this theorem is very similar to the proof of Theorem 4. We only notice that we must observe firstly $S_{r+1}f$ and after that to use (6.3). Namely, from $f \in J'(r)$ we have that $F \in S'_{r+m+1}$ and this implies that we have to observe the dual pair (S'_{r+m+1}, S_{r+m+1}). ($\eta(t)(x+t)^{-r-m-2} \in S_{r+m+1}$ as a function of t.)

7. THE UNIFORM BEHAVIOUR OF $S_r f$

Let F be a continuous function with supp $F \subset [0,\infty)$, $r > \alpha > -1$ and $F(x) \sim x^\alpha$ as $x \to \infty$. Denote by $\Lambda_{a,\varepsilon}$, $a > 0$, $\varepsilon > 0$, a subset of C defined by

$$\Lambda_{a,\varepsilon} = \{a + Re^{i\phi};\ R \geq 0,\ -\pi+\varepsilon \leq \phi \leq \pi-\varepsilon\}.$$

If $z = a + Re^{i\phi} \in \Lambda_{a,\varepsilon}$ and $t \in [0,\infty)$ we have

$$|z + t|^{r+1} > \left(\frac{1-\cos\phi}{2}\right)^{\frac{r+1}{2}} (R + a + t)^{r+1}. \tag{7.1}$$

This follows from the elementary inequalities:

$$(a+t)^2 + 2(a+t)R\cos\phi + R^2 \geq (a+t)^2 - 2(a+t)R\cos\varepsilon + R^2 =$$

$$= (a+t)^2 + R^2 + ((a+t)^2 + R^2)\cos\varepsilon - (a+t+R)^2\cos\varepsilon \geq$$

$$\geq ((a+t)^2 + R^2)(1 + \cos\varepsilon) - 2((a+t)^2 + R^2)\cos\varepsilon = ((a+t)^2 + R^2)(1 - \cos\varepsilon).$$

Assumptions on F imply

$$F(x) < C(1 + x^\alpha),\quad x \geq 0.$$

For $z \in \Lambda_{a,\varepsilon}$ and suitable C_1

$$\left|\int_0^\infty \frac{F(t)}{(z+t)^{r+1}}\,dt\right| \leq C\left(\frac{2}{1-\cos\varepsilon}\right)^{(r+1)/2} \int_0^\infty \frac{(1+t^\alpha)}{(R+a+t)^{r+1}}\,dt \leq C_1\left(\frac{1}{r} + \frac{\Gamma(r-\alpha)\Gamma(\alpha+1)}{\Gamma(r+1)}\right)\left(\frac{1}{a+R}\right)^{r-\alpha}. \tag{7.2}$$

So, we have proved the following lemma:

LEMMA 6. Let F satisfy the conditions given above. The function $z^{r-\alpha}(S_r F)(z)$ is bounded in $\Lambda_{a,\varepsilon}$, $a > 0$, $\varepsilon > 0$.

We use this lemma for the proof of the following Theorem:

THEOREM 7. Let f satisfy the conditions of Theorem 4 and let all the slowly varying functions in Theorem 4 are equal to 1. Then

(i) $$A_{f,r}(z) = ((S_r f)(z) - \sum_{i=1}^{N} c_i \frac{\Gamma(r-\alpha_i)}{\Gamma(r+1)} z^{\alpha_i - r})/z^{\alpha-\ell-r}$$

is a bounded analytic function in any $\Lambda_{a,\varepsilon}$, $a > 0$, $\varepsilon > 0$;

(ii) $A_{f,r}(z)$ converges uniformly to zero in $\Lambda_{a,\varepsilon}$ when $|z| \to \infty$.

PROOF (i). It follows from the structural theorem (3.2) and Lemma 6.

(ii) It follows from (i) and Theorem 4 which enable us to use the Montel Theorem [1, p. 5].

THEOREM 8. Let f satisfy the conditions of Theorem 5 and let all the slowly varying functions in Theorem 5 be equal to 1. Let

$$A_{f,r}(z) = ((S_r f)(z) - \sum_{i=1}^{N} c_i \frac{\Gamma(r-\alpha_i)}{\Gamma(r+1)} z^{\alpha_i - r})/z^{\alpha+\ell-r}.$$

Then

(i) $A_{f,r}(z)$ is a bounded function in $\Lambda_{0,\varepsilon} \cap B(0,R)$, $\varepsilon > 0$, $R > 0$, where $B(0,R) = \{z; |z| < R\}$;

(ii) $A_{f,r}(z)$ converges uniformly to zero in $\Lambda_{0,\varepsilon}$ when $|z| \to 0$.

For the proof of this theorem we need:

LEMMA 9. Let $F \in L^1_{loc}$ supp $F \subset [0,\infty)$, $r > \alpha > -1$, $F(x) \sim x^\alpha$, $x \to 0^+$, and $\int_0^\infty |F(t)(z+t)^{-r-1}|dt < \infty$, $z \in \Lambda_{0,\varepsilon} \cap B(0,R)$. Then $z^{r-\alpha}(S_r F)(z)$ is bounded in $\Lambda_{0,\varepsilon} \cap B(0,R)$, $R > 0$.

PROOF. Take $M > 0$. For suitable C we have

$$\left|\int_0^\infty \frac{F(t)}{(z+t)^{r+1}}\,dt\right| \leq \int_0^M \frac{t^\alpha}{|z+t|^{r+1}}\,dt + \int_M^\infty \frac{|F(t)|}{|z+t|^{r+1}}\,dt.$$

(7.1) implies that

$$\rho^{r-\alpha}\int_M^\infty \frac{|F(t)|dt}{(t+\rho)^{r+1}}, \quad \rho < R.$$

So, it follows that

$$|z^{r-\alpha}| \int_M^\infty \frac{|F(t)|}{|z+t|^{r+1}}\,dt, \quad \rho < R,$$

is bounded, and we have to prove the same for

$$|z^{r-\alpha}| \int_0^M \frac{t^\alpha}{|z+t|^{r+1}}\, dt.$$

On putting $z = \rho\cdot e^{i\phi}$, $0 < \rho < R$, $-\pi+\varepsilon \leqq \phi \leqq \pi-\varepsilon$ we have

$$\int_0^M \frac{t^\alpha}{\rho^{r+1}|e^{i\phi} + \frac{t}{\rho}|^{r+1}}\, dt = \frac{1}{\rho^{r-\alpha}} \int_0^{M/\rho} \frac{u^\alpha du}{(u^2 + 2u\cos\varepsilon + 1)^{(r+1)/2}}$$

$$\leqq \frac{1}{\rho^{r-\alpha}} \int \frac{u^\alpha du}{(u^2 - 2u\cos\varepsilon + 1)^{(r+1)/2}} .$$

This implies the assertion.

PROOF OF THEOREM 8. (i) From the structural theorem (3.2) and Lemma 9 it follows that

$A_{f,r}(z)$ is bounded in $\Lambda_{0,\varepsilon} \cap B(0,R)$.

(ii) Let $\tilde{A}_{f,r}(z) = A_{f,r}(1/z)$, $z \in C \setminus (-\infty,0]$.

The function $\tilde{A}_{f,r}(\omega)$, $\omega \in \Lambda_{0,\varepsilon} \cap \{\omega; |\omega| > 1/R\}$, $\varepsilon > 0$, $R > 0$, is analytic and bounded. As well, we have

$$\tilde{A}_{f,r}(x) \to 0 \text{ as } x \to \infty.$$

This implies that the same assertions hold for $A_{f,r}$ in the domain $\Lambda_{a,\varepsilon} \cap \{\omega;\ |\omega| > 1/R\}$, $a > 0$, $R > 0$. So by the Montel Theorem it follows that $\tilde{A}_{f,r}(z)$ converges uniformly to 0, in $\Lambda_{a,\varepsilon}$ as $|z| \to \infty$. Further on, this implies that $\tilde{A}_{f,r}(z)$ converges uniformly to 0 in $\Lambda_{0,\varepsilon}$ as $|z| \to \infty$ and so, that the assertion (ii) holds.

8. EXAMPLES

By examples 9., 10., 11. and Theorems 4., 5. we have:

12. For $r > 0$ ($r \in R \setminus (-N)$)

$$(S_r(H(t-1)e^{1/t}))(x) - \left(\frac{\Gamma(r)}{\Gamma(r+1)} x^{-r} + x^{-1-r} \ln x\left(-1 + \sum_{r=2}^{\infty}\frac{1}{r!}\,\frac{1}{(r-1)}\right) x^{-1-r}\right)$$
$$= o(x^{-1-r}),\ x \to \infty.$$

13. For $r > -1$

$$(S_r(H(t-1)/t^n))(x) - \left(\frac{1}{(n-1)} x^{-r-1} + \frac{(r+1)}{(n-2)} x^{-r-2} \dots \frac{\Gamma(r+n-1)(-1)^{n-2}}{\Gamma(r+1)(n-2)!} x^{-r-n+1}\right.$$
$$\left. + \frac{\Gamma(r+n)(-1)^{n-1}}{\Gamma(r+1)(n-1)!} x^{-r-n}\right) = o(x^{-n} \ln x),\ x \to \infty.$$

14. For $r > -m$

$$(S_r(H(1-t)t_+^m))(x) - \frac{(-1)^{m-1}}{(m-1)!}\,\frac{\Gamma(r+m)}{\Gamma(r+1)} \left(x^{-m-r} \ln x + \left(\sum_{i=1}^{m-1} \frac{1}{i}\right) x^{-m-r}\right) = o(x^{-m-r}),\ x \to 0^+.$$

15. Let $f \in S_+'$ and have the compact support. It is proved in [4, p. 386] that f has the open q.a.e. at ∞ of order $(\alpha,1)$ with $\alpha \leqq -1$ and of length $-\infty$, i.e. (with suitable $c_i \in C$)

$$f(t) \overset{q.e.}{\sim} \sum^{\infty} c_i f_{\alpha_i+1}(t), \text{ at } \infty,\quad \alpha_i \in N.$$

Corollary 3 and Theorem 4 imply that for $r > \alpha$, $r \in R \setminus (-N)$

$$(S_r f)(x) \sim \sum_{i=1}^{\infty} c_i \frac{\Gamma(r+\alpha_i)}{\Gamma(r+1)} x^{-r-\alpha_i}, \quad x \to \infty.$$

The similar assertion can be formulated for a periodic distribution from S'_+ ([4, p. 386]).

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REFERENCES

1. BOAS, R.P., Entire functions, Academic Press, New York, 1956.
2. CARMICHAEL, R.D., MILTON, E.O., Abelian Theorems for the Distributional Stieltjes Transform, J. Mat. Anal. Appl. 72, (1979), 195-205.
3. McCLURE, J.P., WONG, R., Explicit error terms for asymptotic expansions of Stieltjes transforms, J. Inst. Math. Appl. 22, (1978), 129-145.
4. DROŽINOV, Yu.N., ZAVIALOV, B.I., Quasiasymptotic behaviour of generalized functions and Tauberian theorems in a complex domain, Mat. Sb. 102, (1977), 372-390 (in Russian).
5. GEL'FAND, I.M., SHILOV, G.E., Generalized Functions, V1, Properties and Operations, Acad. Press, New York and London, 1964.
6. LAVOINE, J., MISRA, O.P., Theorèmes abélians pour la transformation de Stieltjes des distributions, C.P. Acad. Sci. Paris 279, (1974), 99-102.
7. LAVOINE, J., MISRA, O.P., Abelian theorems for the distributional Stieltjes transformation, Math. Proc. Cam. Phil. Soc. 86, (1979), 287-293.
8. PILIPOVIĆ, S., An Inversion Theorem for the Distributional Stieltjes Transform, Proc. Edinburgh Math. Soc. 29, (1986), 171-185).
9. PILIPOVIĆ, S., STANKOVIĆ, B., Abelian theorem for the distributional Stieltjes transform Z. Anal. Anvend. 6(4), (1987), 341-349.
10. PILIPOVIĆ, S., STANKOVIĆ, B., Initial value Abelian theoreme for the distributional Stieltjes transform, Studia Math. LXXXXVI, (1987), 239-254.
11. SENETA, E., Regularly varying functions, Lecture Notes in Math., 508, Springer Verlag, Berlin - Heidelberg - New York, 1976.
12. VLADIMIROV, V.S., Generalized Functions in Mathematical Physics, Moscow, 1979.
13. VLADIMIROV, V.S., DROŽINOV, Yu.N., ZAVIALOV, B.I., Several Dimension Tauberian Theorems for Generalized Functions, Nauka, Moscow, 1986 (i. Russian).
14. WONG, R., McClure, J.P., Generalized Mellin convolution and their asymptotic expansions, Can. J. Math. XXXVI, 5, (1984), 924-960.
15. ZAVIALOV, B.I., Automodel asymptotic electromagnetic form factors and the asymptotic behaviour of Fourier transformations in the neighbourhood of the light cone, TMF 17, 2, 178-189 (in Russian).