A NOTE ON WEAKLY QUASI CONTINUOUS FUNCTIONS

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ABSTRACT. In [1], it was shown that a function \( f : X \to Y \) is weakly quasi continuous if and only if \( f^{-1}(\text{Cl}(V)) \subseteq \text{Cl}(\text{Int}(f^{-1}(\text{Cl}(V)))) \) for every open set \( V \) of \( Y \). By utilizing this result, the present author [2] showed that a function \( f : X \to Y \) is weakly quasi continuous if and only if for every regular closed set \( F \) of \( Y \), \( f^{-1}(F) \) is semi-open in \( X \). In this note, the author shows that these results are false and corrects the proofs of Theorem 6.1.7 and Lemma 6.4.4 of [2].

KEY WORDS AND PHRASES. weakly quasi continuous.

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The purpose of this note is to point out that Theorem 2 of [1] and Theorem 4.2 of [2] are false, and to correct the proofs of Theorem 6.1.7 and Lemma 6.4.4 of [2].

Let \( S \) be a subset of a topological space. The closure and the interior of \( S \) are denoted by \( \text{Cl}(S) \) and \( \text{Int}(S) \), respectively. A subset \( S \) is said to be semi-open [3] (resp. regular-closed) if \( S \subseteq \text{Cl}(\text{Int}(S)) \) (resp. \( S = \text{Cl}(\text{Int}(S)) \)). A function \( f : X \to Y \) is said to be semi-continuous [3] (resp. semi-open [4]) if for every open set \( V \) of \( Y \) (resp. \( X \)), \( f^{-1}(V) \) (resp. \( f(V) \)) is semi-open in \( X \) (resp. \( Y \)). A function \( f : X \to Y \) is said to be almost-continuous [5] if for each \( x \in X \) and each open neighborhood \( V \) of \( f(x) \), \( \text{Cl}(f^{-1}(V)) \) is a neighborhood of \( x \).

DEFINITION. A function \( f : X \to Y \) is said to be weakly quasi continuous [1] (briefly \( \text{w.q.c.} \)) if for each \( x \in X \), each open set \( U \) containing \( x \) and each open set \( V \) containing \( f(x) \), there exists an open set \( G \) of \( X \) such that \( \emptyset \neq G \subseteq U \) and \( f(G) \subseteq \text{Cl}(V) \).

Popa and Stan [1] obtained the following characterization of \( \text{w.q.c.} \) functions.

THEOREM A ([1, Theorem 2]). A function \( f : X \to Y \) is \( \text{w.q.c.} \) if and only if \( f^{-1}(\text{Cl}(V)) \subseteq \text{Cl}(\text{Int}(f^{-1}(\text{Cl}(V)))) \) for every open set \( V \) of \( Y \).

In [2], among others, the author established the following three statements.

THEOREM B ([2, Theorem 4.2]). A function \( f : X \to Y \) is \( \text{w.q.c.} \) if and only if for every regular closed set \( F \) of \( Y \), \( f^{-1}(F) \) is semi-open in \( X \).

THEOREM C ([2, Theorem 6.1.7]). The composition \( g \circ f : X \to Z \) of a continuous function \( f : X \to Y \) and a semi-continuous function \( g : Y \to Z \) is not necessarily \( \text{w.q.c.} \).
LEMMA D ([2, Lemma 6.4.4]). Let \( f : X \to Y \) be an open continuous surjection and \( g : Y \to Z \) a function. If \( g \circ f : X \to Z \) is w.q.c., then \( g \) is w.q.c.

The author utilized Theorem A in order to prove Theorem B. Moreover, Theorem B was utilized in the proofs of Theorem C and Lemma D. However, it follows from Example 2 (below) that the necessity of Theorem A is false and hence so is Theorem B. Thus, it is necessary to revise the proofs of Theorem C and Lemma D. For this purpose, we have the following modification of Theorem A.

THEOREM 1. A function \( f : X \to Y \) is w.q.c. if and only if for every open set \( V \) of \( Y \), \( f^{-1}(V) \subseteq \text{Cl}(\text{Int}(f^{-1}(\text{Cl}(V)))) \).

PROOF. Necessity. Suppose that \( f \) is w.q.c. Let \( V \) be any open set of \( Y \) and \( x \in f^{-1}(V) \). For each open set \( G \) of \( X \) containing \( x \), there exists an open set \( U \) of \( X \) such that \( \emptyset \neq U \subseteq G \) and \( f(U) \subseteq \text{Cl}(V) \). Therefore, it follows that \( U \subseteq \text{Cl}(\text{Int}(f^{-1}(\text{Cl}(V)))) \) and \( U \subseteq \text{Int}(f^{-1}(\text{Cl}(V))) \). Since \( \emptyset \neq U \subseteq \text{Int}(f^{-1}(\text{Cl}(V))) \), \( x \in \text{Cl}(\text{Int}(f^{-1}(\text{Cl}(V)))) \) and hence \( f^{-1}(V) \subseteq \text{Cl}(\text{Int}(f^{-1}(\text{Cl}(V)))) \).

Sufficiency. Let \( x \) be any point of \( X \), \( G \) any open set of \( X \) containing \( x \) and \( V \) any open set of \( Y \) containing \( f(x) \). By hypothesis, we have \( x \in f^{-1}(V) \subseteq \text{Cl}(\text{Int}(f^{-1}(\text{Cl}(V)))) \) and hence \( G \cap \text{Int}(f^{-1}(\text{Cl}(V))) \neq \emptyset \). Put \( G \cap \text{Int}(f^{-1}(\text{Cl}(V))) = U \), then we obtain \( \emptyset \neq U \subseteq G \) and \( f(U) \subseteq \text{Cl}(V) \). This shows that \( f \) is w.q.c.

The following example shows that the necessities of Theorems A and B are both false.

EXAMPLE 2. Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, X, \{a\}, \{b, c\}\} \) and \( \sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \). Let \( f : (X, \tau) \to (X, \sigma) \) be the identity function. Then \( f \) is w.q.c. by Theorem 1. Let \( V = \{a\} \in \sigma \), then \( \text{Cl}(V) = \{a, c\} \) is a regular closed set of \( (X, \tau) \). However, \( f^{-1}(\text{Cl}(V)) = \{a, c\} \) and \( \text{Cl}(\text{Int}(f^{-1}(\text{Cl}(V)))) = \{a\} \). Thus, \( f^{-1}(\text{Cl}(V)) \) is not semi-open in \( (X, \tau) \) and \( f^{-1}(\text{Cl}(V)) \notin \text{Cl}(\text{Int}(f^{-1}(\text{Cl}(V)))) \).

REMARK 3. (1) It follows immediately from Theorem 1 that the sufficiencies of Theorems A and B are both true.

(2) In the proof of [2, Theorem 6.1.7], the set \( V = \{b, c, d\} \) is clopen in \( (Z, \emptyset) \), \( (g \circ f)^{-1}(V) = \{b, c, d\} \) and \( \text{Cl}(\text{Int}(g \circ f^{-1}(\text{Cl}(V)))) = \{b\} \). Therefore, by Theorem 1 \( g \circ f \) is not w.q.c. and hence it is not semi-continuous.

Next, we give the correct proof of Lemma D in the improved form.

THEOREM 4. Let \( f : X \to Y \) be a semi-open almost continuous surjection and \( g : Y \to Z \) a function. If \( g \circ f : X \to Z \) is w.q.c., then \( g \) is w.q.c.

PROOF. Let \( W \) be any open set of \( Z \). Since \( g \circ f \) is w.q.c., by Theorem 1 we have \( (g \circ f)^{-1}(W) \subseteq \text{Cl}(\text{Int}(g \circ f^{-1}(\text{Cl}(W)))) \). Since \( f \) is almost continuous, for every subset \( A \) of \( X \), \( f(\text{Cl}(\text{Int}(A))) \subseteq \text{Cl}(f(\text{Int}(A))) \) [6, Theorem 6]. Moreover, since \( f \) is semi-open, \( f(\text{Int}(A)) \subseteq \text{Cl}(\text{Int}(f(A))) \) [4, Theorem 9] and hence \( f(\text{Cl}(\text{Int}(A))) \subseteq \text{Cl}(\text{Int}(f(A))) \) for every subset \( A \) of \( X \). Therefore, we obtain \( g^{-1}(W) \subseteq \text{Cl}(\text{Int}(g^{-1}(\text{Cl}(W)))) \). It follows from Theorem 1 that \( g \) is w.q.c.

THEOREM 5. Let \( f : X \to Y \) be an open continuous surjection. A function \( g : Y \to Z \) is w.q.c. if and only if the composition \( g \circ f : X \to Z \) is w.q.c.

PROOF. Necessity. Let \( W \) be any open set of \( Z \). Since \( g \circ f \) is w.q.c., by Theorem 1 \( g^{-1}(W) \subseteq \text{Cl}(\text{Int}(g^{-1}(\text{Cl}(W)))) \). Since \( f \) is open continuous, for every subset \( B \) of \( Y \), \( f^{-1}(\text{Cl}(\text{Int}(B))) \subseteq \text{Cl}(f^{-1}(B)) \). Therefore, we obtain \( (g \circ f)^{-1}(W) \subseteq \text{Cl}(\text{Int}(g \circ f^{-1}(\text{Cl}(W)))) \) and hence by Theorem 1 \( g \circ f \) is w.q.c.

Sufficiency. Since an open continuous function is semi-open almost continuous, this is an immediate consequence of Theorem 4.
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