1. INTRODUCTION. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be analytic in \( |z| < R \). For a non-decreasing sequence of positive numbers \( \{d_n\} \), the Gelfond-Leontev (G-L) derivative of \( f \) is defined as
\[
Df(z) = \sum_{n=1}^{\infty} d_n a_n z^{n-1}.
\]

The \( k \)th iterate \( D^k f, k=1,2,\ldots \), of \( D \) is given by
\[
D^k f(z) = \sum_{n=k}^{\infty} d_{n-k} \cdots d_1 a_n z^{n-k}
\]
where, \( e_0 = 1 \) and \( e_n = (d_1 d_2 \cdots d_n)^{-1} \), \( n=1,2,\ldots \). If \( d_n \equiv 1 \), \( Df \) is the ordinary derivative of \( f \); whereas, if \( d_n \equiv 1 \), \( D \) is the shift operator \( L \) which transforms
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{into} \quad Lf(z) = \sum_{n=1}^{\infty} a_n z^{n-1}.
\]

Let,
\[
\psi(z) = \sum_{n=0}^{\infty} e_n z^n
\]
and have radius of convergence \( R_0 \). From the monotonicity of \( \{d_n\} \), we have
\[
R_0 = \lim_{n \to \infty} d_n = \sup \{d_n\}.
\]

Clearly, \( \psi(0) = 1 \) and \( D\psi(z) = \psi(z) \). Thus, \( \psi(z) \) bears the same relationship to the operator \( D \) that the function \( \exp(z) \) bears to the ordinary differentiation.

For an entire function \( f \), Nachbin used the function \( \psi(z) \) as a comparison function for measuring the growth of maximum modulus of \( f \) on \( |z| = r \). Thus, the
growth parameter \( \Psi \)-type of \( f \) is defined as the infimum of the positive numbers \( \tau \) such that, for sufficiently large \( r \),

\[
|f(z)| < M\psi(\tau r)
\]

(1.4)

where, \( \psi(z) \) is entire and \( M \) is a positive constant. We denote \( \Psi \)-type of \( f \) as \( \tau_\psi(f) \). It is known \([2,p.6]\) that

\[
\tau_\psi(f) = \lim_{n \to \infty} \sup \left| a_n \right|^{1/n}
\]

(1.5)

For \( d_n \equiv n \), the \( \Psi \)-type of an entire function \( f \) reduces to its classical exponential type and the formula (1.5) gives its well known coefficient characterisation \([3, p.11]\).

The comparison function \( \psi(z) \) can also be used to define a measure of growth analogous to classical order \([3, p.8]\) of an entire function. Thus, for an entire function \( f \), let the \( \Psi \)-order \( \rho_\psi(f) \) of \( f \) be defined as the infimum of positive numbers \( \rho \) such that, for sufficiently large \( r \),

\[
|f(z)| < K\psi(r^\rho)
\]

(1.6)

where \( \psi(z) \) is entire and \( K \) is a positive constant.

Shah and Trimble \([4,5]\) showed that if \( f \) is entire then, the assumption that the classical derivatives \( f^{(p)} \) are univalent in \( \Delta = \{z: |z| < 1\} \) for a suitable increasing sequence \( \{n_p\}_{p=1}^\infty \) of positive integers affects the growth of the maximum modulus of \( f \). If instead, we assume that the G-L derivatives \( D^n_nf \) of an entire function \( f \) are univalent in \( \Delta \), then it is natural to enquire in what way the \( \Psi \)-type and \( \Psi \)-order of \( f \) are influenced. The present paper is an attempt in this direction. In Theorem 1, we find that if \( f \) is entire, \( D^n_nf \) are univalent in \( \Delta \) and

\[
\lim_{p \to \infty} \sup (n_{p+1} - n_p) = \mu, \ 1 < \mu < \infty,
\]

then the \( \Psi \)-type \( \tau_\psi(f) \) of \( f \) must satisfy

\[
\tau_\psi(f) < 2(d(\mu+1) \cdots d(2))^{1/\mu}.
\]

Further, if \( \mu = \infty \), then \( f \) need not be of finite \( \Psi \)-type. Our Theorem 2 shows that if \( f \) is entire, \( D^n_nf \) are univalent in \( \Delta \) and \( n_p \sim n_{p+1} \) as \( p \to \infty \), then

\[
\rho_\psi(f) < \frac{1}{\log d(n_{p+1} - n_p)}
\]

(1.7)

It is clear that if \( 0 < \rho_\psi(f) < 1 \), then the above inequality gives no relationship between \( D^n_nf \) and the \( \Psi \)-order of an entire function \( f \). In fact, no such relation of this nature exists. This is illustrated in Theorem 3, wherein for any given
\(p, 0 < p < 1,\) and any given increasing sequence \(\{n_p\}_{p=1}^{\infty}\) of positive integers, we construct an entire function \(h,\) of \(\psi\)-order \(p,\) such that \(D^n_h\) is univalent in \(\Delta\) if and only if \(n = n_p.\)

In the sequel, we shall assume throughout that \(d_n \to \infty\) as \(n \to \infty.\)

2. \(\psi\)-TYPE AND EXPONENTS OF UNIVALENT \(G-L\) DERIVATIVES.

**Theorem 1.** Let \(f(z) = \sum_{n=0}^{\infty} a_n z^n\) be an entire function and \(\{n_p\}_{p=1}^{\infty}\) be an increasing sequence of positive integers. Let \(D^n f\) be analytic and univalent in \(\Delta.\) Suppose \(\limsup_{p \to \infty} (n_p - n_{p-1}) = \mu, 1 < \mu < \infty.\) Then, the \(\psi\)-type \(T_{\psi}(f)\) of \(f\) satisfies

\[
T_{\psi}(f) < 2(d(\mu+1)\ldots d(2))^{1/\mu}.
\]

**Proof.** By the hypothesis,

\[
D^n f(z) = \sum_{k=0}^{\infty} d(n_p+k)\ldots d(k+1)a(n_p+k)z^k
\]

are univalent in \(\Delta.\) Since, for any function \(G(z) = b_0 + b_1 z + b_2 z^2 + \ldots,\) univalent in \(\Delta,\) it is known [6] that \(|b_n| < n|b_1|\) for \(n = 2, 3, \ldots,\) we get

\[
|a(n_p+k)| < k^{d_k \ldots d_1} d(n_p+1)\ldots d(2) |a(n_p+1)|
\]

for \(k = 1, 2, \ldots\) and \(p = 2, 3, \ldots.\) In particular, putting \(k = n_{p+1}-n_1 + 1,\) and inducting upon \(p,\) we get, for \(p \geq 2\) and \(2 < k < n_{p+1}-n_1 + 1,\)

\[
|a(n_p+k)| < A_k d_k^{d_k \ldots d_1} d(n_p+1)\ldots d(2)
\]

where \(A = d(n_1+1)\ldots d(2)|a(n_1+1)|.\) Hence, for sufficiently large \(p,

\[
\frac{|a(n_p+k)|}{e(n_p+k)} < (1+o(1))(d_k \ldots d_1)^{1/(n_p+k)} (n_{p+1}-n_1 + 1)\ldots d(2) \leq \frac{1/(n_p+k)}{1/(n_{p+1}-n_1 + 1)}
\]

Since, \((d_k \ldots d_1)^{1/(n_p+k)}\) is an increasing function of \(k,\) and

\((n_{p+1}-n_p) < \mu', \mu' > \mu,\) for sufficiently large \(p,

\[
(d_k \ldots d_1)^{1/(n_p+k)} < (d(n_{p+1}-n_p+1)\ldots d(1))^{1/(n_p+1)} (1+o(1))
\]
Further [7], for $p > 2$

$$\sum_{i=1}^{P} \frac{1}{(n_i - n_{i-1} + 1)} \frac{p}{n} < \frac{p}{n} < 2 \quad \text{(2.5)}$$

Using (2.5) and the preceding inequality in (2.4), we get for sufficiently large $p$,

$$\frac{1}{(n + k)} \sum_{p} \frac{a(p + k)}{e(n + k)} < 2(1 + o(1)) \sum_{i=2}^{P} (d(n_i - n_{i-1} + 1) \ldots d(2)) \quad \text{(2.6)}$$

Now, if $a_j > 0$, $t_j > 0$, $\Sigma t_j > 0$ and $\max_{1 \leq j < N-1} \frac{a_j}{N} \leq \frac{a_N}{N}$ then clearly,

$$\frac{1}{\Sigma_{j=1}^{N} a_j} < \frac{a_n}{N} \quad \text{(2.7)}$$

Further, $\log(d(j+1) \ldots d(2))/j$ is an increasing function of $j$ for $1 < j < \nu$, $\nu = 1, 2, \ldots$. Thus, if $1 < j < \nu$,

$$\frac{\log(d(j+1) \ldots d(2))}{j} < \frac{\log(d(\nu+1) \ldots d(2))}{\nu} \quad \text{(2.8)}$$

Let $p > p_o$, $1 < \gamma < \nu$. Suppose $t_\gamma$ is the number of $j_i$'s in $[p_0, p]$ such that

$$n_{j+1} - n_j = \gamma \quad \text{for} \quad j = j_i$$. Then, by (2.7) and (2.8),

$$\frac{p}{p_o + 1} \sum_{j=1}^{p} \frac{\log(d(n_j - n_{j-1} + 1) \ldots d(2))}{n_j - n_{j-1}} < \frac{\sum_{\gamma=1}^{\nu} t_\gamma (\log(d(\gamma+1) \ldots d(2))}{\nu}$$

The above inequality implies that

$$\sum_{i=2}^{P} \frac{1}{(n_i - n_{i-1} + 1)} \ldots d(2) \quad \frac{p}{n} \quad \frac{p}{p_o + 1} \sum_{j=1}^{p} \log(d(n_j - n_{j-1} + 1) \ldots d(2)) \quad \exp \{O(1)\\}$$

$$\leq \exp \{O(1)\\}$$

Using the estimate (2.9) in (2.6) and proceeding to limits

$$\lim_{k \to \infty} \frac{a_k}{e_k} = \lim_{k \to \infty} \left(\frac{a(p + k)}{e(n + k)} \right) \quad \text{for} \quad 2 < k < n_{p+1} - n_{p+1}, \quad p > 2$$

$$\leq 2(d(\nu+1) \ldots d(2))^{1/\nu}$$

This completes the proof of the theorem.
REMARK 1. In Theorem 1, it is sufficient to take the function \( f \) to be analytic in \( |z| < R \), for some \( R > 0 \), if the sequence \( \{d_n\}_{n=1}^{\infty} \) in the definition of the G-L derivative of \( f \) satisfies the condition \( \lim_{m \to \infty} ((\sum_{i=2}^{m} \log(d(i))/m) = \infty \). In fact, for an analytic function \( f \) in \( |z| < R \), if \( D^n f \) are univalent in \( \Delta \),

\[
\lim_{p \to \infty} \sup_{n \to \infty} \left( \frac{n - n_{p-1}}{p} \right) = \mu, \quad 1 < \mu < \infty, \quad \text{and} \quad \lim_{m \to \infty} \left( \sum_{i=2}^{m} \log(d(i))/m \right) = \infty
\]

holds, then \( f \) is necessarily entire. To see this, we use (2.5) and

\[
\left( d_k \ldots d_1 \right)^{1/(n+k)} < 1 + o(1)
\]

for sufficiently large \( p \) in (2.3) to get

\[
\left| a(n+k) \right|^{1/(n+k)} < 2(1+o(1)) \exp \left( \frac{1}{n_p} \sum_{i=2}^{n} \log(d(n_i - n_{i-1} + 1) \ldots d(2)) \right)
\]

for sufficiently large \( p \). But since, for sufficiently large \( p \),

\[
\frac{1}{n_p} \sum_{i=2}^{n} \log(d(n_i - n_{i-1} + 1) \ldots d(2)) \to 0 \quad \text{as} \quad p \to \infty.
\]

Thus, by (2.10) and the condition \( \lim_{m \to \infty} ((\sum_{i=2}^{m} \log(d(i))/m) = \infty \)

\[
\lim_{k \to \infty} \sup_{p \to \infty} \left| a_k \right|^{1/k} = \lim_{p \to \infty} \sup_{k \to \infty} \left| a(n+k) \right|^{1/(n+k)} ; \quad 2 < k < n_{p+1} - n_{p+1}, \quad p > 2
\]

\[
= 0.
\]

REMARK 2. The inequality (2.1) can be improved by imposing suitable additional restrictions on the sequence \( \{d_n\}_{n=1}^{\infty} \). For example, let the sequence \( \{d_n\}_{n=1}^{\infty} \) be such that

\[
\frac{(d(n+2))^n}{d(n+1) \ldots d(2)} > \frac{2}{3(n+1)}, \quad n=1,2,3, \ldots . \tag{2.11}
\]

Note that (2.11) is satisfied for \( d_n = n^\alpha, \alpha > 1 \).

Because of (2.11), the function \( s(j) \) defined by

\[
s(j) = \log(d(j+1) \ldots d(2)) \log(j+1)
\]

is an increasing function of \( j \) and so for \( j=1,2, \ldots \mu; \mu=1,2, \ldots \)
\[
\log(d(j+1)\ldots d(2)) + \log(1) < \log(d(\mu+1)\ldots d(2)) + \log(\mu+1). \tag{2.12}
\]

Let \( t_\gamma \) be the same as in the proof of Theorem 1. Using (2.7) and (2.12), we get

\[
\frac{\sum_{\gamma=1}^{\mu} t_\gamma (\log(d(\gamma+1)\ldots d(2)) + \log(\gamma+1))}{\sum_{\gamma=1}^{\mu} t_\gamma} = \frac{1}{\sum_{\gamma=1}^{\mu} t_\gamma} \sum_{\gamma=1}^{\mu} t_\gamma (\log(d(\gamma+1)\ldots d(2)) + \log(\gamma+1)) < \log(d(\mu+1)\ldots d(2)) + \log(\mu+1). \nonumber
\]

Again, we have

\[
\frac{\prod_{p=1}^{\mu} \left( \log(d(j+1)\ldots d(2)) + \log(j+1) \right)}{\prod_{p=1}^{\mu} \log(j+1)} < \frac{\prod_{p=1}^{\mu} t_\gamma (\log(d(\gamma+1)\ldots d(2)) + \log(\gamma+1))}{\prod_{p=1}^{\mu} t_\gamma} = \frac{1}{\prod_{p=1}^{\mu} t_\gamma} \sum_{\gamma=1}^{\mu} t_\gamma (\log(d(\gamma+1)\ldots d(2)) + \log(\gamma+1)).
\]

The above inequality, when employed in (2.4), gives

\[
\frac{1}{(n+k)} \frac{\log(d(n^1-n_i-1+1)\ldots d(2))}{\log(1+O(1))} \frac{1}{(n+k)} < \exp \left\{ \frac{\log(d(n^1-n_i-1+1)\ldots d(2))}{\log(n^1-n_i-1+1)} \right\}.
\]

Now, on proceeding to limits, we get

\[
\tau_\psi(f) < (\mu+1)^{1/\mu}(d(\mu+1)\ldots d(2))^{1/\mu}. \tag{2.13}
\]

It is clear that the bound on \( \tau_\psi(f) \) in (2.13) is better than that in (2.1).

**Remark 3.** By taking \( \mu=1 \), Theorem 1 gives \( \tau_\psi(f) < 2d(2) \), a result recently proved in [8].

Theorem 1 shows that if \( (n-p_n) = O(1) \), then \( f \) is of finite \( \psi \)-type.

We now give an example to show that if \( \lim_{p \to \infty} \sup_{p} (n_p-n_{p-1}) = \infty \), then \( f \) need not be of finite \( \psi \)-type.

**Example.** Let \( \{n_p\}_{p=1}^{\infty} \) be an increasing sequence of positive integers such that \( (n_{p+1}-n_p) > 2 \) for all \( p \). Further, assume that the sequence \( \{d_n\}_{n=1}^{\infty} \) is such that

1. \( d_1 = 1 \) and \( \log d(n) \sim \log n \) as \( n \to \infty \)
2. \( n_p = o(n) \)
3. \( \eta_p = o(n \log d(n)) \)
where, $n_p = \sum_{i=2}^{P} \log(d(n_i - n_{i-1} + 1)) \ldots d(2)$.

Let $\psi$ be a non-decreasing step function such that $\psi(n_1) = \psi(n_2)$, and

$$\psi(n_p) = \exp\left(\frac{n_p}{p}\right), \quad p > 2$$

and

$$\psi(x) = \psi(n_p), \quad n_p < x < n_{p+1}.$$ 

Let

$$g_{j+1}(z) = \begin{cases} \frac{\psi(j)}{d(j+1) \ldots d(2) (j - n_p + 1)} & \text{if } j = n_p \text{ for some } p \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$g(z) = \sum_{j=0}^{\infty} g_j z^j.$$

We first show that $g$ is an entire function. We have

$$\limsup_{k \to \infty} \frac{1}{k} \sum \psi(n_p) \frac{1}{d(n_p+1) \ldots d(2)} = \frac{\exp(n_p/n_p)}{p+1} \limsup_{p \to \infty} \exp\left(\frac{n_p}{p}\right).$$

Since $\log d(n) \sim \log n$ as $n \to \infty$, using the condition (iii), we get from the above inequality that

$$\limsup_{k \to \infty} \frac{|g_k|^{1/k}}{k} = 0.$$

Hence $g$ is entire. It is easily seen that $g$ is of order 1. But, by the condition (ii),

$$\limsup_{k \to \infty} \frac{|g_k|^{1/k}}{k} = \limsup_{p \to \infty} \exp\left(\frac{n_p}{p}\right) = \limsup_{p \to \infty} \exp\left(\frac{n_p}{2p}\right) = \infty.$$

Thus, $f$ is not of finite $\psi$-type. It remains to see that

$$D^p g(z) = \sum_{k=1}^{\infty} d(n_{p+k}+1) \ldots d(n_{p+k}-n_{p+2}) a(n_{p+k}+1) z^{n_{p+k}-n_{p+1}}$$

are univalent in $\Delta$. To this end, it is enough to prove that

$$\sum_{k=1}^{\infty} \frac{d(n_{p+k}+1) \ldots d(2)}{d(n_{p+k}-n_{p+1}) \ldots d(2)} |a(n_{p+k}+1)|.$$
or, equivalently to show that
\[ \psi(n) < \psi(n + 1) \]

Using the definition of \( \psi \), the last inequality reads as
\[ \prod_{k=1}^{p+k} \left( \frac{\exp(n - np - 1)}{d(n-k-np+1)\ldots d(2)} \right) < 1. \quad (2.14) \]

Now, an induction on \( k \), gives, for \( k = 1, 2, 3, \ldots \)
\[ \exp(n - np - 1) \prod_{k=1}^{p} \left( \frac{\exp(n - np - 1)}{d(n-k-np+1)\ldots d(2)} \right) < 1. \]

Hence, (2.14) is clearly satisfied.

### 3. \( \psi \)-ORDER AND EXPONENTS OF UNIVALENT G-L DERIVATIVES.

A function \( S(x) \), continuous on \( [1, \infty) \), is said to be Slowly Oscillating (S.O.) if for every positive number \( c > 0 \),
\[ \lim_{x \to \infty} \frac{S(cx)}{S(x)} = 1. \]

A function \( H(n) \) is said to be the restriction of a Slowly Oscillating function \( S(x) \) if \( S(n) = H(n) \) for every positive integer \( n \). It is known [9] that, as \( k \to \infty \)
\[ \sum_{i=1}^{k} H(i) \sim kH(k). \quad (3.1) \]

**THEOREM 2.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function of \( \psi \)- order \( \rho_\psi \) and \( \{n_p\}_{p=1}^{\infty} \) be a strictly increasing sequence of positive integers. Let \( D_{np} f \) be analytic and univalent in \( \Delta \), such that \( n_p \sim np+1 \) as \( p \to \infty \). If \( \log d(n) \) is the restriction of a slowly oscillating function on integers, then
\[ \rho_\psi(f) \leq \frac{1}{1-\lim_{p \to \infty} \sup \frac{\log d(n-p-1)}{\log d(n)}}. \quad (3.2) \]

We need the following lemmas.

**LEMMA 1.** Let \( \gamma \) be defined by (1.3). Let \( \gamma_n = \min_{x>0} \psi(x^n)x^{-n} \), \( a > 0 \).

Then,
\[ \gamma_n \leq e^{\frac{n(1 - \frac{1}{a})}{a}} \frac{a(n+a)}{a}. \quad (3.3) \]

**PROOF.** Since \( \{d_n\}_{n=1}^{\infty} \) is increasing, we note that for any pair of integers \( k \) and \( n \), \( e_k \leq e^{n-k} \). Thus,
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\[ \psi(x^a) = \sum_{k=0}^\infty \frac{x^a}{k!} \]

Let \( 0 < w < 1 \). Setting \( x = w^{1/a} \), we get

\[ \psi(x^a)_w^{1-n} \leq e \frac{d_n}{n} \left( \frac{1}{1-w^a} \right)^n. \]

Choosing \( w = (n/n+a)^{1/a} \) to minimize the right-hand side of the above inequality, we have

\[ \gamma_n \leq \min_{0 < w < 1} \psi(x^a)_w^{1-n} \leq e \frac{d_n}{n} \left( \frac{(n+a)}{a} \right)^n. \]

**Lemma 2.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function of \( \psi \)-order \( \rho \),

where the sequence \( \{d(n)\} \) in \( D_f \) is such that \( \log d(n) \) is the restriction of a slowly oscillating function on positive integers.

Then,

\[ \rho \psi(f) = \lim sup_{n \to \infty} \frac{n \log d(n)}{\log |a_n|}. \tag{3.4} \]

**Proof.** By Cauchy's inequality, we get

\[ |a_n| \leq M(r) r^{-n}, \quad M(r) = \max_{|z| < r} |f(z)|. \]

Since \( f \) is of \( \psi \)-order \( \rho \), \( \rho \psi(f) = \rho \), for any \( \varepsilon > 0 \), \( |f(z)| < M \psi(r^{\rho+\varepsilon}). \)

So that

\[ |a_n| < M \psi(r^{\rho+\varepsilon}) r^{-n}. \]

Using Lemma 1, we have

\[ |a_n| < M e d_n \left( \frac{1}{\rho+\varepsilon} \right)^n. \tag{3.5} \]

But, since \( \log d(n) \) is the restriction of a S.O. function, by (3.1),

\[ \log d(n) \sim n \log d(n) \text{ as } n \to \infty. \]

Thus, it follows from (3.5)

\[ \lim sup_{n \to \infty} \frac{n \log d(n)}{-\log |a_n|} < \rho. \]

To prove that equality holds in (3.4), suppose that

\[ \lim sup_{n \to \infty} \frac{n \log d(n)}{-\log |a_n|} < \rho. \]

Then, there exist \( p_1 < p \) such that \( |a_n| < e_{n_{01}} r^{n_{01}} \) for \( n > n_0 \). It now follows that, for \( |z| = r \),

\[ |f(z)| < \sum_{n=0}^{n_0} |a_n| r^n + \sum_{n=n_0+1}^{\infty} |a_n| r^n \]

\[ < 0(1) + \sum_{n=n_0+1}^{\infty} e_{n_{01}} r^n. \tag{3.6} \]
Choose

$$N(r) = \frac{\log \psi(r)}{\log r}.$$  

It is easily seen that $N(r) \to \infty$ as $r \to \infty$. Since for all values of $k$ and $n$, $e_n < d_n^{k-n}$, we have

$$\sum_{n=0}^{\infty} \frac{1}{\rho_1} e_n r^n < \sum_{n=0}^{\infty} \frac{1}{\rho_1} r^{n-\rho_1} d_k r^n = \frac{1}{\rho_1} e_k n \sum_{n=0}^{\infty} \left( \frac{r}{d_k} \right)^n.$$  

Let $k$ be chosen such that $(r/d_k) < 1$. Then,

$$\sum_{n=0}^{\infty} \frac{1}{\rho_1} e_n r^n < \frac{k+1/\rho_1}{d_k} e_k 1/\rho_1 \left( \frac{r}{d_k} \right).$$  

Since the left hand side of (3.7) is independent of $k$, letting $k \to \infty$, we get

$$\sum_{n=0}^{\infty} \frac{1}{\rho_1} r^n < 1.$$  

Thus

$$\sum_{n=N(r)}^{\infty} \frac{1}{\rho_1} e_n r^n = o(1), \text{ as } r \to \infty.$$  

Since, $r^{N(r)} = \exp(N(r) \log r) = \psi(r^{\rho_1})$, it now follows from (3.6)

$$|f(z)| < O(1) + \sum_{n=0}^{\infty} \frac{1}{\rho_1} e_n r^n + o(1)$$

$$< O(1) \psi(r^{\rho_1}).$$  

Since $\rho_1 < \rho$ and $\rho$ is the $\psi$-order of $f$, the above inequality contradicts the definition of $\psi$-order. Thus, equality must hold in (3.4). This proves the lemma.

**PROOF OF THEOREM 2.** Since $D$ are univalent in $\Delta$, from (2.2), we get for sufficiently large $p$ and $2k < p+1 - n_p + 1$.

$$|a(n + k)|^{1/(n + k)} < (1 + o(1)) \left( \frac{\cdots d_1}{d_k \cdots d_1} \right) \frac{p}{p+1 - n_p + 1} \frac{1}{p!} \left( \begin{array}{c} n_p - 1 \\ n_p - 1 \\ \vdots \\ n_p - 1 \\ n_p - 1 + 1 \end{array} \right) d(1) \cdots d(2)$$

Further, we have
Using these inequalities, (2.5) and (3.8), it follows that, for sufficiently large $p$,

$$\left| a(n_{p+k}) \right| \leq \frac{1}{n_{p+k}} \frac{1}{p^{+1}}$$

Let,

$$M_p = \max \{ \log d(n_i - n_{i-1} + 1) : 2 \leq i \leq p \}.$$

Since $\log d(n)$ is the restriction of a slowly oscillating function on integers, by (3.1)

$$\log d(n_{p}) - \frac{\sum_{i=2}^{n_p} \log d(n_i - n_{i-1} + 1)}{n_p} \leq \frac{\sum_{i=2}^{n_p} \log d(n_i - n_{i-1} + 1)}{n_p} - \log d(n_{p+1}).$$

Consequently, for sufficiently large $p$,

$$\frac{(n_{p+k}) \log d(n_{p+k})}{-\log |a(n_{p+k})|} < \frac{\log d(n_{p+1})}{\log d(n_p) - \frac{n_{p+1}}{n_p} M_{p+1} \log 2}.$$

Again, from the definition of S.O. function $\log d(n_p) \sim \log d(n_{p+1})$ as $p \to \infty$.

Hence,

$$\rho_\psi < \frac{1}{1 - \limsup_{p \to \infty} \frac{M_p}{\log d(n_p)}}.$$

If $M_p$ is bounded, there is nothing to prove. So, let $M_p = \infty$ as $p \to \infty$.

For $p > 2$, let,

$$A_p = \frac{\log d(n_p - n_{p-1} + 1)}{\log d(n_p)}$$

and

$$B_p = \frac{M_p}{\log d(n_p)}.$$

But as $M_p = \max \{ \log d(n_i - n_{i-1} + 1) : 2 \leq i \leq p \}$, for each $p > 2$, there is some
Let $p_n < p < q_p$ such that $M_p = \log d(n_p - n_{p-1})$. Hence

$$B_p < A_p.$$ Taking $q_p \to \infty$,

$$\limsup_{p \to \infty} B_p \leq \limsup_{p \to \infty} A_p.$$ 

Now (3.2) follows from (3.10).

**COROLLARY.** Suppose the conditions of Theorem 2 are satisfied. If as $p \to \infty$,

$$\log d(n_p - n_{p-1}) = o(\log d(n_p))$$

then,

$$\rho_{\psi}(f) < 1.$$ 

**THEOREM 3.** Let $0 < \rho < 1$. Let $\{n_p\}_{p=1}^\infty$ be a strictly increasing sequence of non-negative integers. Then, there is an entire function $h$ of $\psi$-order $\rho$ such that $D^n h$ is univalent in $\Delta$ if and only if $n = n_p$ for some $p$.

**PROOF.** Suppose $\rho > 0$ and $\{d(n_p)\}_{n=1}^\infty$ is an increasing sequence of positive numbers such that $\log d(n_p)$ is the restriction of a slowly oscillating function on integers and $d_1 = 1$. Let,

$$h_{j+1} = \begin{cases} \frac{1}{2^p d(n_p + 1) \ldots d(2) (j - n_p + 1)} & \text{if } j = n_p \\ 0 & \text{otherwise.} \end{cases}$$

Define, $h(z) = \sum_{j=0}^\infty h_j z^j$. Then, $h(z)$ is an entire function and

$$\rho_{\psi}(h) = \limsup_{k \to \infty} \frac{k \log d(k)}{-\log |h_k|}$$

$$= \limsup_{p \to \infty} \frac{(n+1) \log d(n+1)}{p \log 2 + \frac{1}{\rho} \log (d(n_p+1) \ldots d(2))} = \rho.$$ 

To show that $D^n h$ given by

$$D^n h(z) = \sum_{k=0}^\infty \frac{d(n_p + k + 1) \ldots d(2)}{d(n_p + k - n_p + 1) \ldots d(2)} h(n_p + k + 1) z^{n_p + k - n_p + 1}$$

is univalent in $\Delta$, it is enough to prove that

$$\sum_{k=1}^\infty \frac{d(n_p + k + 1) \ldots d(2)}{d(n_p + k - n_p + 1) \ldots d(2)} |h(n_p + k + 1)|$$

$$< d(n_p + 1) \ldots d(2) |h(n_p + 1)|.$$
Since \( p < 1 \),

\[
\sum_{k=1}^{n} \frac{(n+k_p-1)\ldots d(2)}{d(n+k_p-n_p+1)\ldots d(2)} \left| h(n+k_p+1) \right|
\]

\[
< \frac{1}{2^p} \sum_{k=1}^{n} \frac{(n+k_p-1)\ldots d(2)}{d(n+k_p-n_p+1)\ldots d(2)} \left( 1 - \frac{1}{\rho} \right)
\]

\[
< \frac{1}{2^p} \left( d(n+1)\ldots d(2) \right) \sum_{k=1}^{n} \frac{1}{2^k}
\]

\[
= d(n+1)\ldots d(2) \left| h(n+1) \right|
\]

As \( D_{n+1}h(0) = 0 \) unless \( n=n_p \) for some \( p \), only \( D_{n_p}^{n}h \) are univalent in \( \Delta \).

If \( p=0 \), then take \( h_{j+1}^{*} \) defined by

\[
h_{j+1}^{*} = \left\{ \begin{array}{ll}
\frac{1}{2^p d(n+1)\ldots d(2)} & \text{if } j=n_p \text{ for some } p, \\
0 & \text{otherwise.}
\end{array} \right.
\]

in place of \( h_{j+1} \) in the Taylor series of the function \( h(z) \).

REFERENCES


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