GROWTH OF ENTIRE FUNCTIONS WITH SOME UNIVALENT GELFOND-LEONTEV DERIVATIVES

G.P. KAPOOR, O.P. JUNEJA and J. PATEL

Department of Mathematics
Indian Institute of Technology, Kanpur
Kanpur, 208016, India

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1. INTRODUCTION. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $|z| < R$. For a non-decreasing sequence of positive numbers $\{d_n\}_{n=1}^{\infty}$, the Gelfond-Leontev (G-L) derivative of f is defined as [1]

$$Df(z) = \sum_{n=1}^{\infty} d_n a_n z^{n-1}. \quad (1.1)$$

The k th iterate $D^k f$, $k=1,2,\dots$, of D is given by

$$D^k f(z) = \sum_{n=k}^{\infty} d_n \dots d_{n-k+1} a_n z^{n-k} \quad (1.2)$$

$$= \sum_{n=k}^{\infty} \frac{e_{n-k}}{e_n} a_n z^{n-k}$$

where, $e_0=1$ and $e_n = (d_1 d_2 \dots d_n)^{-1}$, $n=1,2,\dots$. If $d_n \equiv n$, Df is the ordinary derivative of f ; whereas, if $d_n \equiv 1$, D is the shift operator L which transforms

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ into } Lf(z) = \sum_{n=1}^{\infty} a_n z^{n-1}.$$

Let,

$$\psi(z) = \sum_{n=0}^{\infty} e_n z^n \quad (1.3)$$

and have radius of convergence R_0 . From the monotonicity of $\{d_n\}_{n=1}^{\infty}$, we have

$$R_0 = \lim_{n \rightarrow \infty} d_n = \sup_{n \geq 1} \{d_n\}.$$

Clearly, $\psi(0) = 1$ and $D\psi(z) = \psi(z)$. Thus, $\psi(z)$ bears the same relationship to the operator D that the function $\exp(z)$ bears to the ordinary differentiation.

For an entire function f , Nachbin used the function $\psi(z)$ as a comparison function for measuring the growth of maximum modulus of f on $|z| = r$. Thus, the

growth parameter ψ -type of f is defined as the infimum of the positive numbers τ such that, for sufficiently large r ,

$$|f(z)| < M\psi(\tau r) \quad (1.4)$$

where, $\psi(z)$ is entire and M is a positive constant. We denote ψ -type of f as $\tau_\psi(f)$. It is known [2, p.6] that

$$\tau_\psi(f) = \lim_{n \rightarrow \infty} \sup \left| \frac{a_n}{e^n} \right|^{1/n}. \quad (1.5)$$

For $d_n \equiv n$, the ψ -type of an entire function f reduces to its classical exponential type and the formula (1.5) gives its well known coefficient characterisation [3, p. 11].

The comparison function $\psi(z)$ can also be used to define a measure of growth analogous to classical order [3, p.8] of an entire function. Thus, for an entire function f , let the ψ -order $\rho_\psi(f)$ of f be defined as the infimum of positive numbers ρ such that, for sufficiently large r ,

$$|f(z)| < K\psi(r^\rho) \quad (1.6)$$

where $\psi(z)$ is entire and K is a positive constant.

Shah and Trimble [4,5] showed that if f is entire then, the assumption that the classical derivatives $f^{(n)}_p$ are univalent in $\Delta = \{z: |z| < 1\}$ for a suitable increasing sequence $\{n_p\}_{p=1}^\infty$ of positive integers affects the growth of the maximum modulus of f . If instead, we assume that the G-L derivatives $D^{n_p}_p f$ of an entire function f are univalent in Δ , then it is natural to enquire in what way the ψ -type and ψ -order of f are influenced. The present paper is an attempt in this direction. In Theorem 1, we find that if f is entire, $D^{n_p}_p f$ are univalent in Δ and $\lim_{p \rightarrow \infty} \sup (n_p - n_{p-1}) = \mu$, $1 < \mu < \infty$, then the ψ -type $\tau_\psi(f)$ of f must satisfy

$$\tau_\psi(f) < 2(d(\mu+1) \dots d(2))^{1/\mu}.$$

Further, if $\mu = \infty$, then f need not be of finite ψ -type. Our Theorem 2 shows that

if f is entire, $D^{n_p}_p f$ are univalent in Δ and $n_p \sim n_{p+1}$ as $p \rightarrow \infty$, then

$$\rho_\psi(f) < \frac{1}{1 - \lim_{p \rightarrow \infty} \sup \frac{\log d(n_p - n_{p-1})}{\log d(n_p)}}.$$

It is clear that if $0 < \rho_\psi(f) < 1$, then the above inequality gives no relationship between $D^{n_p}_p f$ and the ψ -order of an entire function f . In fact, no such relation of this nature exists. This is illustrated in Theorem 3, wherein for any given

ρ , $0 < \rho < 1$, and any given increasing sequence $\{n_p\}_{p=1}^{\infty}$ of positive integers, we construct an entire function h , of ψ -order ρ , such that $D^{n_p} h$ is univalent in Δ if and only if $n=n_p$.

In the sequel, we shall assume throughout that $d_n \rightarrow \infty$ as $n \rightarrow \infty$.

2. ψ -TYPE AND EXPONENTS OF UNIVALENT G-L DERIVATIVES.

THEOREM 1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function and $\{n_p\}_{p=1}^{\infty}$ be an increasing sequence of positive integers. Let $D^{n_p} f$ be analytic and univalent in Δ . Suppose $\lim_{p \rightarrow \infty} \sup (n_p - n_{p-1}) = \mu$, $1 < \mu < \infty$. Then, the ψ -type $\tau_{\psi}(f)$ of f satisfies

$$\tau_{\psi}(f) \leq 2(d(\mu+1) \dots d(2))^{1/\mu}. \quad (2.1)$$

PROOF. By the hypothesis,

$$D^{n_p} f(z) = \sum_{k=0}^{\infty} d(n_p+k) \dots d(k+1) a_{n_p+k} z^k$$

are univalent in Δ . Since, for any function $G(z) = b_0 + b_1 z + b_2 z^2 + \dots$, univalent in Δ , it is known [6] that $|b_n| \leq n|b_1|$ for $n=2,3,\dots$, we get

$$|a_{n_p+k}| \leq k \frac{d_k \dots d_1}{d_{k+n_p} \dots d_1} d(n_p+1) \dots d(2) |a_{n_p+1}| \quad (2.2)$$

for $k=1,2,\dots$ and $p=2,3,\dots$. In particular, putting $k=n_{p+1}-n_p+1$ and inducting upon p , we get, for $p \geq 2$ and $2 \leq k \leq n_{p+1}-n_p+1$,

$$|a_{n_p+k}| \leq A k \frac{d_k \dots d_1}{d_{k+n_p} \dots d_1} \prod_{i=2}^p (n_i - n_{i-1} + 1) d(n_i - n_{i-1} + 1) \dots d(2) \quad (2.3)$$

where $A = d(n_1+1) \dots d(2) |a_{n_1+1}|$. Hence, for sufficiently large p ,

$$\begin{aligned} & \left| \frac{a_{n_p+k}}{e_{n_p+k}} \right|^{1/(n_p+k)} \\ & \leq (1+o(1)) (d_k \dots d_1)^{1/(n_p+k)} \prod_{i=2}^p \{(n_i - n_{i-1} + 1) d(n_i - n_{i-1} + 1) \dots d(2)\}^{1/(n_p+k)} \end{aligned} \quad (2.4)$$

Since, $(d_k \dots d_1)^{1/(n_p+k)}$ is an increasing function of k , and

$(n_{p+1} - n_p) \leq \mu'$, $\mu' > \mu$, for sufficiently large p ,

$$(d_k \dots d_1)^{1/(n_p+k)} \leq (d(n_{p+1} - n_p + 1) \dots d(1))^{1/n_{p+1}} = (1+o(1))$$

Further [7], for $p > 2$

$$\prod_{i=2}^p (n_i - n_{i-1} + 1)^{1/(n_p + 2)} < (1 + \frac{n_p}{p})^{p/n_p} < 2. \quad (2.5)$$

Using (2.5) and the preceding inequality in (2.4), we get for sufficiently large p ,

$$\left| \frac{a(n_p + k)}{e(n_p + k)} \right|^{1/(n_p + k)} < 2(1 + o(1)) \prod_{i=2}^p (d(n_i - n_{i-1} + 1) \dots d(2))^{1/(n_p + k)} \quad (2.6)$$

Now, if $a_j > 0$, $t_j > 0$, $\sum t_j > 0$ and $\max_{1 \leq j \leq N-1} (\frac{a_j}{j}) < \frac{a_N}{N}$ then clearly,

$$\frac{\sum_{j=1}^N a_j t_j}{\sum_{j=1}^N j t_j} < \frac{a_N}{N}. \quad (2.7)$$

Further, $\log(d(j+1) \dots d(2))/j$ is an increasing function of j for

$1 < j < \mu$, $\mu = 1, 2, \dots$. Thus, if $1 < j < \mu$,

$$\frac{\log(d(j+1) \dots d(2))}{j} < \frac{\log(d(\mu+1) \dots d(2))}{\mu} \quad (2.8)$$

Let $p > p_0$, $1 < \gamma < \mu$. Suppose t_γ is the number of j_1 's in $[p_0, p]$ such that

$n_{j+1} - n_j = \gamma$ for $j = j_1$. Then, by (2.7) and (2.8),

$$\frac{\sum_{j=p_0+1}^p \log(d(n_j - n_{j-1} + 1) \dots d(2))}{\sum_{j=p_0+1}^p (n_j - n_{j-1})} = \frac{\sum_{\gamma=1}^{\mu} t_\gamma (\log(d(\gamma+1) \dots d(2)))}{\sum_{\gamma=1}^{\mu} \gamma t_\gamma} < \frac{\log(d(\mu+1) \dots d(2))}{\mu}.$$

The above inequality implies that

$$\begin{aligned} \prod_{i=2}^p (d(n_i - n_{i-1} + 1) \dots d(2))^{1/(n_p + k)} &< \exp \left\{ \frac{\sum_{i=2}^p \log(d(n_i - n_{i-1} + 1) \dots d(2))}{n_p} \right\} \\ &< \exp \left\{ o(1) + \frac{\sum_{j=p_0+1}^p \log(d(n_j - n_{j-1} + 1) \dots d(2))}{\sum_{j=p_0+1}^p (n_j - n_{j-1})} \right\} \\ &< \exp \left\{ o(1) + \frac{\log(d(\mu+1) \dots d(2))}{\mu} \right\}. \end{aligned} \quad (2.9)$$

Using the estimate (2.9) in (2.6) and proceeding to limits

$$\begin{aligned} \lim_{k \rightarrow \infty} \sup \left| \frac{a_k}{e_k} \right|^{1/k} &= \lim \sup \left\{ \left| \frac{a(n_p + k)}{e(n_p + k)} \right|^{1/(n_p + k)} : 2 \leq k \leq n_{p+1} - n_p + 1, p > 2 \right\} \\ &< 2(d(\mu+1) \dots d(2))^{1/\mu}. \end{aligned}$$

This completes the proof of the theorem.

REMARK 1. In Theorem 1, it is sufficient to take the function f to be analytic in $|z| < R$, for some R , $0 < R < \infty$, if the sequence $\{d_n\}_{n=1}^{\infty}$ in the definition of G-L derivative of f satisfies the condition $\lim_{m \rightarrow \infty} ((\sum_{i=2}^m \log d(i))/m) = \infty$. In fact, for an analytic function f in $|z| < R$, if $D_p^n f$ are univalent in Δ ,

$$\lim_{p \rightarrow \infty} \sup (n_p - n_{p-1}) = \mu, \quad 1 \leq \mu < \infty, \text{ and}$$

$$\lim_{m \rightarrow \infty} \frac{\sum_{i=2}^m \log d(i)}{m} = \infty$$

holds, then f is necessarily entire. To see this, we use (2.5) and

$$(d_k \dots d_1)^{1/(n_p + k)} \leq 1 + o(1)$$

for sufficiently large p in (2.3) to get

$$\begin{aligned} & |a(n+k)|^{1/(n_p + k)} \\ & \leq 2(1+o(1)) \exp \left[\frac{1}{n_p} \sum_{i=2}^{n_p} \log(d(n_i - n_{i-1} + 1) \dots d(2)) \right. \\ & \quad \left. - \frac{1}{n_p + k} \sum_{i=2}^{n_p + k} \log d(i) \right] \end{aligned} \quad (2.10)$$

for sufficiently large p . But since, for sufficiently large p , $(n_p - n_{p-1}) \leq \mu', \mu' > \mu$,

$$\frac{1}{n_p} \sum_{i=2}^{n_p} \log(d(n_i - n_{i-1} + 1) \dots d(2)) \rightarrow 0 \text{ as } p \rightarrow \infty.$$

Thus, by (2.10) and the condition $\lim_{m \rightarrow \infty} ((\sum_{i=2}^m \log d(i))/m) = \infty$

$$\begin{aligned} \lim_{k \rightarrow \infty} \sup |a_k|^{1/k} &= \lim_{k \rightarrow \infty} \sup \{ |a(n_p + k)|^{1/(n_p + k)} : 2 \leq k \leq n_{p+1} - n_p + 1, p \geq 2 \} \\ &= 0. \end{aligned}$$

REMARK 2. The inequality (2.1) can be improved by imposing suitable additional restrictions on the sequence $\{d_n\}_{n=1}^{\infty}$. For example, let the sequence $\{d_n\}_{n=1}^{\infty}$ be such that

$$\frac{\{d(n+2)\}^n}{d(n+1) \dots d(2)} > \frac{2}{3}(n+1), \quad n=1, 2, 3, \dots \quad (2.11)$$

Note that (2.11) is satisfied for $d_n = n^\alpha, \alpha > 1$.

Because of (2.11), the function $s(j)$ defined by

$$s(j) = \frac{\log(d(j+1) \dots d(2)) + \log(j+1)}{j}$$

is an increasing function of j and so for $j=1, 2, \dots, \mu; \mu=1, 2, \dots$

$$\frac{\log(d(j+1)\dots d(2))+\log(j+1)}{j} \leq \frac{\log(d(\mu+1)\dots d(2))+\log(\mu+1)}{\mu}. \quad (2.12)$$

Let t_γ be the same as in the proof of Theorem 1. Using (2.7) and (2.12), we get

$$\frac{\prod_{j=0}^p \{\log(d(j+1)\dots d(2))+\log(j+1)\}}{\prod_{j=0}^p (n_j - n_{j-1})} = \frac{\prod_{\gamma=1}^{\mu} t_\gamma \{\log(d(\gamma+1)\dots d(2))+\log(\gamma+1)\}}{\prod_{\gamma=1}^{\mu} \gamma t_\gamma} \leq \frac{\log(d(\mu+1)\dots d(2))+\log(\mu+1)}{\mu}.$$

Again, we have

$$\prod_{i=2}^p \{(n_i - n_{i-1} + 1)d(n_i - n_{i-1} + 1)\dots d(2)\}^{1/(n_p + k)} \leq \exp \{O(1) + \frac{\prod_{j=0}^p \{\log(d(n_j - n_{j-1} + 1)\dots d(2)) + \log(n_j - n_{j-1} + 1)\}}{\prod_{j=0}^p (n_j - n_{j-1})}\}.$$

The above inequality, when employed in (2.4), gives

$$\left| \frac{a(n_p + k)}{e(n_p + k)} \right|^{1/(n_p + k)} \leq (1+O(1)) \prod_{i=2}^p \{(n_i - n_{i-1} + 1)d(n_i - n_{i-1} + 1)\dots d(2)\}^{1/(n_p + k)} \leq (\mu+1)^{1/\mu} (d(\mu+1)\dots d(2))^{1/\mu}.$$

Now, on proceeding to limits, we get

$$\tau_\psi(f) \leq (\mu+1)^{1/\mu} (d(\mu+1)\dots d(2))^{1/\mu}. \quad (2.13)$$

It is clear that the bound on $\tau_\psi(f)$ in (2.13) is better than that in (2.1).

REMARK 3. By taking $\mu=1$, Theorem 1 gives $\tau_\psi(f) \leq 2d(2)$, a result recently proved in [8].

Theorem 1 shows that if $(n_p - n_{p-1}) = O(1)$, then f is of finite ψ -type. We now give an example to show that if $\lim_{p \rightarrow \infty} \sup (n_p - n_{p-1}) = \infty$, then f need not be of finite ψ -type.

EXAMPLE. Let $\{n_p\}_{p=1}^\infty$ be an increasing sequence of positive integers such that $(n_{p+1} - n_p) > 2$ for all p . Further, assume that the sequence $\{d_n\}_{n=1}^\infty$ is such that

$$(i) \quad d_1 \equiv 1 \text{ and } \log d(n) \sim \log n \text{ as } n \rightarrow \infty$$

$$(ii) \quad n_p = o(n_p)$$

$$(iii) \quad n_p = o(n_p \log d(n_p))$$

where, $\eta_p = \sum_{i=2}^p \log(d(n_i - n_{i-1} + 1) \dots d(2))$,

Let ψ be a non-decreasing step function such that $\psi(n_1) = \psi(n_2)$,

$$\psi(n_p) = \frac{\exp(\eta_p)}{2^{p-1}}, \quad p \geq 2$$

and

$$\psi(x) = \psi(n_p) \quad n_p < x \leq n_{p+1}.$$

Let

$$g_{j+1} = \begin{cases} \frac{\psi(j)}{d(j+1) \dots d(2) (j - n_p + 1)} & \text{if } j = n_p \text{ for some } p \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$g(z) = \sum_{j=0}^{\infty} g_j z^j$$

We first show that g is an entire function. We have

$$\begin{aligned} \lim_{k \rightarrow \infty} \sup |g_k|^{1/k} &= \lim_{p \rightarrow \infty} \sup \left[\frac{\psi(n_p)}{d(n_p+1) \dots d(2)} \right]^{1/n_p+1} \\ &< \lim_{p \rightarrow \infty} \sup \left[\frac{\exp(\eta_p/n_p)}{(d(n_p+1) \dots d(2))^{1/n_p+1}} \right] \\ &= \lim_{p \rightarrow \infty} \sup \left[\exp \left(\frac{\eta_p}{n_p} - \frac{1}{n_p+1} \sum_{i=2}^{n_p+1} \log d(i) \right) \right]. \end{aligned}$$

Since $\log d(n) \sim \log n$ as $n \rightarrow \infty$, using the condition (iii), we get from the above inequality that

$$\lim_{k \rightarrow \infty} \sup |g_k|^{1/k} = 0.$$

Hence g is entire. It is easily seen that g is of order 1. But, by the condition (ii),

$$\begin{aligned} \lim_{k \rightarrow \infty} \sup \left| \frac{g_k}{e_k} \right|^{1/k} &= \lim_{p \rightarrow \infty} \sup \left(\frac{\psi(n_p)}{e(n_p+1)d(n_p+1) \dots d(2)} \right)^{1/n_p+1} \\ &> \lim_{p \rightarrow \infty} \sup \frac{\exp(\eta_p/n_p)}{2} = \infty. \end{aligned}$$

Thus, f is not of finite ψ -type. It remains to see that

$$D^n_p g(z) = \sum_{k=1}^n d(n_{p+k}+1) \dots d(n_{p+k}-n_p+2) a(n_{p+k}+1) z^{n_{p+k}-n_p+1}$$

are univalent in Δ . To this end, it is enough to prove that

$$\sum_{k=1}^{\infty} (n_{p+k}-n_p+1) \frac{d(n_{p+k}+1) \dots d(2)}{d(n_{p+k}-n_p+1) \dots d(2)} |a(n_{p+k}+1)|$$

$$< d(n_p+1) \dots d(2) |a(n_p+1)|;$$

or, equivalently to show that

$$\sum_{k=1}^{\infty} \frac{\psi(n_{p+k})}{d(n_{p+k}-n_p+1) \dots d(2)} < \psi(n_p).$$

Using the definition of ψ , the last inequality reads as

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\exp(\eta_{p+k}-\eta_p)}{d(n_{p+k}-n_p+1) \dots d(2)} < 1. \quad (2.14)$$

Now, an induction on k , gives, for $k=1, 2, 3, \dots$

$$\exp(\eta_{p+k}-\eta_p) = \prod_{i=1}^{p+k} d(n_i - n_{i-1} + 1) \dots d(2) < d(n_{p+k}-n_p+1) \dots d(2)$$

Hence, (2.14) is clearly satisfied.

3. Ψ -ORDER AND EXPONENTS OF UNIVALENT G-L DERIVATIVES.

A function $S(x)$, continuous on $[1, \infty)$, is said to be Slowly Oscillating (S.O.) if for every positive number $c > 0$,

$$\lim_{x \rightarrow \infty} \frac{S(cx)}{S(x)} = 1.$$

A function $H(n)$ is said to be the restriction of a Slowly Oscillating function $S(x)$ if $S(n) = H(n)$ for every positive integer n . It is known [9] that, as $k \rightarrow \infty$

$$\sum_{i=1}^k H(i) \sim kH(k). \quad (3.1)$$

THEOREM 2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of ψ -order ρ_{ψ} and $\{n_p\}_{p=1}^{\infty}$ be a strictly increasing sequence of positive integers. Let $D^{n_p} f$ be analytic and univalent in Δ , such that $n_p \sim n_{p+1}$ as $p \rightarrow \infty$. If $\log d(n)$ is the restriction of a slowly oscillating function on integers, then

$$\rho_{\psi}(f) < \frac{1}{1 - \limsup_{p \rightarrow \infty} \left[\frac{\log d(n_p - n_{p-1})}{\log d(n_p)} \right]}. \quad (3.2)$$

We need the following lemmas.

LEMMA 1. Let ψ be defined by (1.3). Let $\gamma_n = \min_{x>0} \psi(x^a) x^{-n}$, $a > 0$. Then,

$$\gamma_n < e_n d_n^{n(1 - \frac{1}{a})} \left(\frac{e(n+a)}{a} \right). \quad (3.3)$$

PROOF. Since $\{d_n\}_{n=1}^{\infty}$ is increasing, we note that for any pair of integers k and n , $e_k < e_n d_n^{n-k}$. Thus,

$$\psi(x^a) = \sum_{k=0}^{\infty} e_k x^{ak} < e_n d_n^n \sum_{k=0}^{\infty} d_n^{-k} x^{ak}.$$

Let $0 < w < 1$. Setting $x_w = wd_n^{1/a}$, we get

$$\psi(x_w^a) x_w^{-n} < e_n d_n^{n(1 - \frac{1}{a})} \frac{w^{-n}}{(1-w^a)}.$$

Choosing $w = (n/(n+a))^{1/a}$ to minimize the right-hand side of the above inequality, we have

$$\gamma_n < \min_{0 < w < 1} \psi(x_w^a) x_w^{-n} < e_n d_n^{n(1 - \frac{1}{a})} \left(\frac{e(n+a)}{a} \right).$$

LEMMA 2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of ψ -order ρ_ψ , where the sequence $\{d(n)\}$ in Df is such that $\log d(n)$ is the restriction of a slowly oscillating function on positive integers.

Then,

$$\rho_\psi(f) = \limsup_{n \rightarrow \infty} \frac{n \log d(n)}{-\log |a_n|}. \quad (3.4)$$

PROOF. By Cauchy's inequality, we get

$$|a_n| < M(r) r^{-n}, \quad M(r) = \max_{|z| \leq r} |f(z)|.$$

Since f is of ψ -order $\rho_\psi(f) \equiv \rho$, for any $\epsilon > 0$, $|f(z)| < M\psi(r^{\rho+\epsilon})$.

So that

$$|a_n| < M\psi(r^{\rho+\epsilon}) r^{-n}.$$

Using Lemma 1, we have

$$|a_n| < M e_n d_n^{n(1 - \frac{1}{\rho+\epsilon})} \left(\frac{e(n+\rho+\epsilon)}{(\rho+\epsilon)} \right). \quad (3.5)$$

But, since $\log d(n)$ is the restriction of a S.O. function, by (3.1),

$$\sum_{i=2}^n \log d(i) \sim n \log d(n) \text{ as } n \rightarrow \infty. \text{ Thus, it follows from (3.5)}$$

$$\limsup_{n \rightarrow \infty} \frac{n \log d(n)}{-\log |a_n|} < \rho.$$

To prove that equality holds in (3.4), suppose that

$$\limsup_{n \rightarrow \infty} \frac{n \log d(n)}{-\log |a_n|} < \rho.$$

Then, there exist $\rho_1 < \rho$ such that $|a_n| < e_n^{1/\rho_1}$ for $n > n_0$. It now follows that, for $|z| = r$,

$$|f(z)| < \sum_{n=0}^{n_0} |a_n| r^n + \sum_{n_0+1}^{\infty} |a_n| r^n \quad (3.6)$$

$$< O(1) + \sum_{n_0+1}^{\infty} e_n^{1/\rho_1} r^n.$$

Choose

$$N(r) = \frac{\log \psi(r^{\rho_1})}{\log r}.$$

It is easily seen that $N(r) \rightarrow \infty$ as $r \rightarrow \infty$. Since for all values of k and n , $e_n < e_k d_k^{k-n}$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} e_n^{1/\rho_1} r^n &< \sum_{n=0}^{\infty} e_k^{1/\rho_1} d_k^{k-n/\rho_1} r^n \\ &= d_k^{1/\rho_1} e_k^{1/\rho_1} \sum_{n=0}^{\infty} \left(\frac{r}{d_k^{1/\rho_1}}\right)^n. \end{aligned}$$

Let k be chosen such that $(r/d_k^{1/\rho_1}) < 1$. Then,

$$\sum_{n=0}^{\infty} e_n^{1/\rho_1} r^n < \frac{d_k^{k+1/\rho_1} e_k^{1/\rho_1}}{(d_k^{1/\rho_1} - r)}. \quad (3.7)$$

Since the left hand side of (3.7) is independent of k , letting $k \rightarrow \infty$, we get

$$\sum_{n=0}^{\infty} e_n^{1/\rho_1} r^n < 1.$$

Thus

$$\sum_{n=N(r)}^{\infty} e_n^{1/\rho_1} r^n = o(1), \text{ as } r \rightarrow \infty.$$

Since, $r^{N(r)} = \exp(N(r) \log r) = \psi(r^{\rho_1})$, it now follows from (3.6)

$$|f(z)| < o(1) + \sum_{n=0}^{N(r)} e_n^{1/\rho_1} r^n + o(1)$$

$$< o(1) \psi(r^{\rho_1}).$$

Since $\rho_1 < \rho$ and ρ is the ψ -order of f , the above inequality contradicts the definition of ψ -order. Thus, equality must hold in (3.4). This proves the lemma.

PROOF OF THEOREM 2. Since $D_p^n f$ are univalent in Δ , from (2.2), we get for sufficiently large p and $2 < k < n_{p+1} - n_p + 1$.

$$\begin{aligned} |a(n+k)_p| &\leq \frac{1}{p^{n+k}} \\ &< (1 + o(1)) \left(\frac{d_k \dots d_1}{d_{k+n} \dots d_1} \right)^{1/(n+k)} \prod_{i=2}^p \{(n_i - n_{i-1} + 1) d(n_i - n_{i-1} + 1) \dots d(2)\}^{1/(n+k)} \end{aligned} \quad (3.8)$$

Further, we have

$$(d_k \dots d_1)^{1/(n_p+k)} \leq (d(n_{p+1}-n_p+1) \dots d(1))^{1/n_{p+1}}$$

and

$$(d_{n_p+k} \dots d_1)^{-1/(n_p+k)} \leq (d(n_p+2) \dots d(1))^{-1/(n_p+2)}.$$

Using these inequalities, (2.5) and (3.8), it follows that, for sufficiently large p ,

$$\begin{aligned} & |a(n_p+k)|^{1/(n_p+k)} \\ & \leq \frac{2(1+o(1))}{(d(n_p) \dots d(1))^{1/n_p}} \prod_{i=2}^{p+1} (d(n_i - n_{i-1} + 1))^{(n_i - n_{i-1})/n_p} \end{aligned} \quad (3.9)$$

Let,

$$M_p = \max \{ \log d(n_i - n_{i-1} + 1) : 2 \leq i \leq p \}.$$

Since $\log d(n)$ is the restriction of a slowly oscillating function on integers, by (3.1)

$$\begin{aligned} & \log \frac{\prod_{i=2}^{p+1} d(n_i - n_{i-1} + 1)^{(n_i - n_{i-1})/n_p}}{(d(n_p) \dots d(1))^{1/n_p}} \\ & \leq \frac{1}{n_p} \left[\sum_{i=2}^{p+1} (n_i - n_{i-1}) \log d(n_i - n_{i-1} + 1) - \sum_{i=1}^{n_p} \log d(i) \right] \\ & \leq \frac{n_{p+1}}{n_p} M_{p+1} - \log d(n_p). \end{aligned}$$

Consequently, for sufficiently large p ,

$$\frac{(n_p + k) \log d(n_p + k)}{-\log |a(n_p + k)|} \leq \frac{\log d(n_{p+1} + 1)}{\log d(n_p) - \frac{n_{p+1}}{n_p} M_{p+1} - \log 2}$$

Again, from the definition of S.O. function $\log d(n_p) \sim \log d(n_{p+1})$ as $p \rightarrow \infty$.

Hence,

$$\rho_\psi \leq \frac{1}{1 - \limsup_{p \rightarrow \infty} \frac{M_p}{\log d(n_p)}}. \quad (3.10)$$

If M_p is bounded, there is nothing to prove. So, let $M_p \rightarrow \infty$ as $p \rightarrow \infty$.

For $p \geq 2$, let,

$$A_p = \frac{\log d(n_p - n_{p-1} + 1)}{\log d(n_p)}$$

and

$$B_p = \frac{M_p}{\log d(n_p)}.$$

But as $M_p = \max \{ \log d(n_i - n_{i-1} + 1) : 2 \leq i \leq p \}$, for each $p \geq 2$, there is some

$q_p, q_p < p$ such that $M_p = \log d(n_{q_p} - n_{q_p-1} + 1)$. Hence

$B_p < A_{q_p}$. Taking $q_p \rightarrow \infty$,

$$\limsup_{p \rightarrow \infty} B_p < \limsup_{p \rightarrow \infty} A_p.$$

Now (3.2) follows from (3.10).

COROLLARY. Suppose the conditions of Theorem 2 are satisfied. If as $p \rightarrow \infty$,

$$\log d(n_p - n_{p-1}) = o(\log d(n_p))$$

then,

$$\rho_\psi(f) < 1.$$

THEOREM 3. Let $0 < \rho < 1$. Let $\{n_p\}_{p=1}^\infty$ be a strictly increasing sequence of non-negative integers. Then, there is an entire function h of ψ -order ρ such that $D^n h$ is univalent in Δ if and only if $n = n_p$ for some p .

PROOF. Suppose $\rho > 0$ and $\{d_n\}_{n=1}^\infty$ is an increasing sequence of positive numbers such that $\log d(n)$ is the restriction of a slowly oscillating function on integers and $d_1 = 1$. Let,

$$h_{j+1} = \begin{cases} \frac{1}{2^p d(n_p+1) \dots d(2) (j - n_p + 1)} & \text{if } j = n_p \text{ for some } p \\ 0 & \text{otherwise.} \end{cases}$$

Define, $h(z) = \sum_{j=0}^\infty h_j z^j$. Then, $h(z)$ is an entire function and

$$\begin{aligned} \rho_\psi(h) &= \limsup_{k \rightarrow \infty} \frac{k \log d(k)}{-\log |h_k|} \\ &= \limsup_{p \rightarrow \infty} \frac{(n_p+1) \log d(n_p+1)}{p \log 2 + \frac{1}{p} \log(d(n_p+1) \dots d(2))} = \rho. \end{aligned}$$

To show that $D^{n_p} h$ given by

$$D^{n_p} h(z) = \sum_{k=0}^\infty (n_{p+k} - n_p + 1) \frac{d(n_{p+k}+1) \dots d(2)}{d(n_{p+k} - n_p + 1) \dots d(2)} h(n_{p+k}+1) z^{n_{p+k} - n_p + 1}$$

is univalent in Δ , it is enough to prove that

$$\begin{aligned} \sum_{k=1}^\infty (n_{p+k} - n_p + 1) \frac{d(n_{p+k}+1) \dots d(2)}{d(n_{p+k} - n_p + 1) \dots d(2)} |h(n_{p+k}+1)| \\ < d(n_p+1) \dots d(2) |h(n_p+1)|. \end{aligned}$$

Since $\rho < 1$,

$$\begin{aligned} & \sum_{k=1}^{\infty} \binom{n_{p+k}-n_{p+1}}{n_{p+k}-n_{p+1}} \frac{d(n_{p+k}+1) \dots d(2)}{d(n_{p+k}-n_{p+1}+1) \dots d(2)} |h(n_{p+k}+1)| \\ & < \frac{1}{2^p} \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{(d(n_{p+k}+1) \dots d(2))^{1-\frac{1}{\rho}}}{d(n_{p+k}-n_{p+1}+1) \dots d(2)} \\ & < \frac{1}{2^p} (d(n_{p+1}) \dots d(2))^{1-\frac{1}{\rho}} \sum_{k=1}^{\infty} \frac{1}{2^k} \\ & = d(n_{p+1}) \dots d(2) |h(n_{p+1})|. \end{aligned}$$

As $D^{n+1}h(0) = 0$ unless $n=n_p$ for some p , only $D^{n_p}h$ are univalent in Δ .

If $\rho=0$, then take h_{j+1}^* defined by

$$h_{j+1}^* = \left\{ \begin{array}{ll} \frac{1}{2^{p+d(n_p+1) \dots d(2)} (j-n_p+1)} & \text{if } j=n_p \text{ for some } p. \\ 0 & \text{otherwise.} \end{array} \right\}$$

in place of h_{j+1} in the Taylor series of the function $h(z)$.

REFERENCES

1. GELFOND, A.O. and LEONTEV, A.F. On a generalization of Fourier series, Mat. Sb. (N.S.) **29** (71), (1951), 477-500.
2. BOAS, R.P., BUCK, R.C. Polynomial expansions of analytic functions, Ergebnisse der Mathematik und ihrer Grenzgebiete, **19**, Springer-Verlag, 1964.
3. BOAS, R.P. Entire Functions, Academic Press, 1954.
4. SHAH, S.M. and TRIMBLE, S.Y. The order of an entire function with some derivatives univalent, J. Math. Anal. Appl. **46** (1974), 395-409.
5. SHAH, S.M. Analytic functions with some derivatives univalent and a related conjecture, Atti. Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **61** fasc. 5 (1976), 344-353.
6. BRANGES, L.De. A proof of Bieberbach Conjecture, Acta. Math. **154** (1985), 137-152.
7. SHAH, S.M. and TRIMBLE, S.Y. Entire functions with some derivatives univalent, Canadian J. Math. **26** (1974), 207-213.

8. JUNEJA, O.P. and SHAH, S.M. Univalent functions with univalent Gelfond-Leontev derivatives, Bull. Austr. Math. Soc. 29 (1984), 329-348.
9. HARDY, G.H. and ROGOSINSKI, W.W. Note on Fourier series (III), Asymptotic formulae for the sums of certain trigonometrical series, Quart. J. Math. (Oxford Ser.), 16 (1954), 49-58.