RESEARCH NOTES

A NOTE ON GLOBAL EXISTENCE FOR BOUNDARY VALUE PROBLEMS

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ABSTRACT. Upper and lower solutions are used in establishing global existence results for certain two-point boundary value problems for \( y''' = f(x, y, y', y'') \) and \( y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) \).

KEY WORDS AND PHRASES. Boundary value problem, global existence.

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1. INTRODUCTION.

In this paper, we will be concerned primarily with the global existence of solutions of boundary value problems for the third order ordinary differential equation

\[ y''' = f(x, y, y', y''), \quad (1.1) \]

satisfying boundary conditions of the form

\[ y(a) = y_1, \quad y'(a) = y_2, \quad y'(b) = y_3, \quad a < b. \quad (1.2) \]

The result we obtain for (1.1), (1.2) is an extension, in some sense, of those for boundary value problems for second order equations which appeared in a recent paper by Umamaheswaram and Subasini [1]. The results in [1] made use of, or were compared to, results dealing with upper and lower solutions for second order equations obtained by Jackson and Schrader [2], Lees [3], and Schrader [4-6]. In [1, Theorem 1], the following is proved.

THEOREM 1.1. Assume that with respect to the second order equation,

\[ y'' = g(x, y, y'), \]

the following are satisfied:

(A.I) \( g: [a, \beta] \times \mathbb{R}^2 \to \mathbb{R} \) is continuous.

(B.I) Solutions of initial value problems exist on \([a, \beta]\) or become unbounded.

(C.I) There exists a sequence \( \{M_j\} \) of real numbers \( +\infty \), such that \( f(x, M_j, 0) > 0 \), for every \( j \geq 1 \) and all \( a < x < \beta \).

(D.I) There exists a sequence \( \{N_j\} \) of real numbers \( -\infty \), such that \( f(x, N_j, 0) < 0 \), for every \( j \geq 1 \) and all \( a < x < \beta \).

Then the boundary value problem

\[ y''' = g(x, y, y'), \]

\[ y(x_1) = y_1, \quad y(x_2) = y_2, \]

where \( a < x_1 < x_2 < \beta \), and \( y_1, y_2 \in \mathbb{R} \), has a solution.

In Section 2, we extend Theorem 1.1 to boundary value problems (1.1), (1.2).
For this extension, we generalize (C.1) and (D.1) so that the conditions set forth by Klaasen [7] for (1.1), (1.2) are satisfied for any $y_i \in \mathbb{R}$, $i=1, 2, 3$.

In Section 3, the results we obtained for (1.1), (1.2) are generalized somewhat to boundary value problems for the $n$th order equation

$$y^{(n)}(a) = f(x, y, y', ..., y^{(n-1)}), \quad (1.3)$$

satisfying

$$y^{(i-1)}(a) = y_i, \quad 1 \leq i \leq n-1, \quad y^{(n-2)}(b) = y_n, \quad a < b. \quad (1.4)$$

We conclude Section 3 with an example.

**2. GLOBAL EXISTENCE FOR (1.1), (1.2).**

In this section, a theorem is proved concerning the global existence of solutions of (1.1), (1.2). We assume in this section that with respect to (1.1), the following are satisfied.

(A.2) $f(x, u_1, u_2, u_3): [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous.

(B.2) Solutions of initial value problems for (1.1) extend to $[a, b]$ or become unbounded.

(C.2) There exist sequences $(L_j)$ and $(M_j)$ of real numbers with both $L_j + \to \infty$ and $M_j + \to \infty$, such that $f(x, M_j x + L_j, M_j, 0) > 0$, for all $i, j \geq 1$ and all $a < x < b$.

(D.2) There exist sequences $(K_j)$ and $(N_j)$ of real numbers, with both $K_j + \to \infty$ and $N_j + \to \infty$, such that $f(x, N_j x + K_j, N_j, 0) < 0$, for all $i, j \geq 1$ and all $a < x < b$.

**THEOREM 2.1.** Assume that (A.2) - (D.2) are satisfied and that $f(x, u_1, u_2, u_3)$ is nonincreasing in $u_1$ for each fixed $x, u_2, u_3$. Then the boundary value problem (1.1), (1.2) has a solution for any choice of $y_1, y_2, y_3 \in \mathbb{R}$.

**PROOF.** Let $y_1, y_2, y_3 \in \mathbb{R}$ be given. By hypotheses (C.2) and (D.2), there exist $I, J \in \mathbb{N}$ such that

$$N_j a + K_I \leq y_1 \leq M_j a + L_I,$$

and

$$N_j \leq \min \{y_2, y_3\} \leq \max \{y_2, y_3\} \leq M_j.$$  

Defining $\gamma(x) = N_j x + K_I$ and $\psi(x) = M_j x + L_I$, it follows $\gamma^{(i)}(x) \leq \psi^{(i)}(x)$ on $[a, b]$, for $i = 0, 1$. Furthermore, by (C.2) and (D.2), $\gamma(x)$ and $\psi(x)$ are lower and upper solutions, respectively, of (1.1) on $[a, b]$.

It follows from results due to Klaasen [7] that there exists a solution $y(x)$ of (1.1), (1.2), for this choice of $y_1, y_2, y_3$, and furthermore $\gamma(x) \leq y(x) \leq \psi(x)$ and $N_j \leq y^{(i)}(x) \leq M_j$ on $[a, b]$. The proof is complete.

**3. GLOBAL EXISTENCE FOR (1.3), (1.4).**

In this section, we will be concerned with the existence of solutions of (1.3), (1.4). For this consideration, results due to Kelley [8] will be used. We assume here that with respect to (1.3), the following are satisfied.

(A.3) $f(x, u_1, u_2, ..., u_n): [a, b] \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

(B.3) Solutions of initial value problems for (1.3) extend to $[a, b]$ or become unbounded.

(C.3) There exist sequences $(M_{1,j})$, $(M_{2,j})$, ..., $(M_{n-1,j})$ of real numbers, such that $M_{k,j} + \to \infty$, $1 \leq k \leq n-1$, and such that if $p_{1,j_2}^{j_1} ... j_{n-1}(x) = \sum_{k=1}^{n-1} M_{k,j_k} x^k$,
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then \( f(x, p_j, \ldots, j_{n-1}(x), p_{n-j}^2, \ldots, j_{n-1}(x), 0) \geq 0 \), for all \( j_1, \ldots, j_{n-1} \geq 1 \) and all \( a \leq x \leq b \).

(D.3) There exist sequences \( \{N_{k,j} \}, \{N_{k+1,j} \}, \ldots, \{N_{n-1,j} \} \) of real numbers, such that \( \sum_{k=1}^{n-k} N_{k, j} x^k \), then \( f(x, q_j, \ldots, j_{n-1}(x), q_{n-2}^2, \ldots, j_{n-1}(x), 0) \leq 0 \), for all \( j_1, \ldots, j_{n-1} \geq 1 \) and all \( a \leq x \leq b \).

THEOREM 3.1. Assume in addition to conditions (A.3) - (D.3) that, if \( y(x) \) is a solution of (1.3) with maximal interval of existence \( I \subseteq [a, b] \), such that \( y^{(n-2)}(x) \) is bounded on \( I \), then \( y^{(n-1)}(x) \) is bounded on \( I \). Furthermore, assume that for each \( 1 \leq i \leq n-2 \), \( f(x, u_1, u_2, \ldots, u_n) \) is nonincreasing in \( u_i \) for each fixed \( x, u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n \). Then the boundary value problem (1.3), (1.4) has a solution for any choice of \( y_i \in \mathbb{R} \), \( 1 \leq i \leq n \).

PROOF. Let \( y_i \in \mathbb{R} \), \( 1 \leq i \leq n \), be given. It follows from (C.3) and (D.3) that there exist \( j_1, j_2, \ldots, j_{n-1} \in \mathbb{N} \) such that

\[
q_{(n-2)}(a) \leq y_1 \leq q_{(n-1)}(a), 1 \leq i \leq n-2,
\]

\[
q_{(n-2)}(a) = \min \{y_{(n-1)}, y_n\} \leq \max \{y_{(n-1)}, y_n\} = p_{(n-2)}(a).
\]

Defining \( \gamma(x) \equiv q_{j_1} \ldots j_{n-1}(x) \) and \( \psi(x) \equiv p_{j_1} \ldots j_{n-1}(x) \), it follows from

\[
\psi^{(n-2)}(x) - \gamma^{(n-2)}(x) = \psi^{(n-2)}(a) - \gamma^{(n-2)}(a) > 0,
\]

for all \( a \leq x \leq b \), and from \( \psi^{(n-1)}(a) > \gamma^{(n-1)}(a) \), \( 1 \leq i \leq n-2 \), that \( \gamma^{(n-1)}(x) < \psi^{(n-1)}(x) \) on \([a, b] \), for \( 1 \leq i \leq n-2 \). Furthermore, from (C.3) and (D.3), \( \gamma(x) \) and \( \psi(x) \) are lower and upper solutions, respectively, of (1.3) on \([a, b] \). It follows from the other hypotheses of the Theorem and from a result due to Kelley [8] that there exists a solution \( y(x) \) of (1.3), (1.4), for this choice of \( y_i \in \mathbb{R} \), \( 1 \leq i \leq n \). Moreover, \( \gamma^{(n-1)}(x) \leq y^{(n-1)}(x) \leq \psi^{(n-1)}(x) \) on \([a, b] \), for \( 1 \leq i \leq n-1 \). This completes the proof.

EXAMPLE. Let \( g: \mathbb{R} \rightarrow \mathbb{R} \) be defined by

\[
g(u) = \begin{cases} 
\sin \left( \frac{u}{e^n} \right), & u \leq 0, \\
-2u, & 0 \leq u \leq e^n, \\
-2e^n + 2e^n \sin \left( \frac{u}{e^n} \right), & u \geq e^n,
\end{cases}
\]

and let \( f(x, u_1, \ldots, u_n): [0, \pi] \times \mathbb{R}^n \rightarrow \mathbb{R} \) be defined by

\[
f(x, u_1, \ldots, u_{n-1}, u_n) = g(u_{n-1}) + 2u_n.
\]

The conditions of Theorem 3.1 are satisfied with respect to the differential equation

\[
y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) = g(y^{(n-2)}) + 2y^{(n-1)}.
\]

In particular, \( \frac{\partial f}{\partial u_i} \), \( 1 \leq i \leq n \), are piecewise continuous and bounded on \([0, \pi] \times \mathbb{R}^n \), hence initial value problems of (3.1) exist on \([0, \pi] \). Also, the sequences \( \{M_{k,j} \} = \{j+1 \} \), for \( 1 \leq k \leq n-2 \), and \( \{N_{n-1,j} \} = \{n/2 \} \) satisfy condition (C.3), whereas, the sequences \( \{M_{k,j} \} = \{-j \} \), for \( 1 \leq k \leq n-2 \), and \( \{N_{n-1,j} \} = \{-j/e^n \} \) satisfy condition (D.3). Hence, by Theorem 3.1, boundary value problems for (3.1)
satisfying
\[ y^{(1-1)}(0) = y_1^*, \quad 1 \leq 1 \leq n-1, \quad y^{(n-2)}(\pi) = y_n \]
are solvable.
In fact,
\[
y(x) = \begin{cases}
\frac{C}{(-4)^k} \left[ e^x \sin x + \sum_{i=1}^{k} (-1)^i \left[ 2^{21-1} \left( \frac{x^{4i-1}}{(4i-1)!} + \frac{x^{4i-2}}{(4i-2)!} \right) \\
+ 2^{21-2} \frac{x^{4i-3}}{(4i-3)!} \right] \right], & n = 4k + 2, \ k = 0, 1, 2, \ldots, \\
\frac{C}{2(-4)^k} \left[ e^x (\sin x - \cos x) + \sum_{i=1}^{k} (-1)^i \left[ 2^{21-1} \left( \frac{x^{4i+1}}{(4i+1)!} + \frac{x^{4i}}{(4i)!} \right) \\
+ 2^{21-1} \frac{x^{4i-1}}{(4i-1)!} \right] + 1 \right], & n = 4k + 3, \ k = 0, 1, 2, \ldots, \\
\frac{C}{(-4)^k} \left[ -e^x \cos x + \sum_{i=1}^{k} (-1)^i \left[ 2^{21-1} \left( \frac{x^{4i+2}}{(4i+2)!} + \frac{x^{4i-3}}{(4i-3)!} \right) \\
+ 2^{21-2} \frac{x^{4i-4}}{(4i-4)!} \right] \right], & n = 4k+1, \ k = 1, 2, \ldots,
\end{cases}
\]
where \(0 < C < 1\), are infinitely many solutions of (3.1) satisfying
\[ y^{(1-1)}(0) = y^{(n-2)}(\pi) = 0, \quad 1 \leq 1 \leq n-1.\]

REFERENCES
1. UMAMAHESWARAM, S. and SUHASINI, M.V.S. A global existence theorem for the boundary value problems of \(y'' = f(x, y, y')\), Nonlinear Anal. 10 (1986), 679-681.