ABSTRACT. Let $f: [0,1] \to [0,1]$ be a piecewise linear map having an infinite number of partition points. Consider $f$ restricted to the domain $D_N = \{ \frac{a}{(m-1)p^N}, (a,p) = 1 \}$, where $p$ is a prime number. The main result establishes an explicit bound for the number of periodic orbits of $f|_{D_N}$, namely $A p^\beta N$, where $A$ and $\beta$ are constants.

KEY WORDS: Periodic Orbits, Transformation of an Interval.

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1. Introduction.

Since the results of Sarkovskii's famous paper [1] have become known, there has been a great deal of research done on the periodic points of continuous maps of an interval into itself. Recently, Hofbauer [2] has generalized some of these results to piecewise monotonic transformations.

In [3,4] maps with a countable number of monotonic segments were studied and conditions were given establishing the existence of absolute continuous invariant measures. The specification property [5], satisfied by many maps, guarantees that any invariant measure (in particular the absolutely continuous one) can be approached by the measures supported on periodic orbits. This motivates the study of periodic orbits for maps with an infinite number of partition points.

In this note we study a class of piecewise linear transformations with an infinite number of partition points. It is shown that if such a transformation is restricted to certain domains, then an explicit bound can be obtained for the number of periodic orbits in that domain. This has practical application in determining the distribution of long periodic orbits [6], and is, in turn, related to the study of computer orbits [7].

Let $m$ be an integer $> 1$. We define the piecewise linear map $f: [0,1] \to [0,1]$ on the intervals...
$I_h = \left[ \frac{h}{m-1}, \frac{h+1}{m-1} \right]$, $h = 0, 1, 2, \ldots, m-2$, as follows: first we define $f$ on $I_0$; we partition $I_0$ by

$$a_0^{(0)} = 0, \quad a_i^{(0)} = \sum_{j=1}^{i} \frac{j}{m};$$

note that $\lim_{i \to \infty} a_i^{(0)} = \frac{1}{m-1}$. We define $f(a_0^{(0)}) = 0, f(a_1^{(0)}) = 1, f(a_2^{(0)}) = 0, \ldots$.

Thus $f$ is piecewise linear and continuous on $I_0$, with slopes $+m, -m^2, m^3, \ldots$, on $[0, \frac{1}{m}]$,

$$\left[ \frac{1}{m}, \frac{1}{m^2}, \frac{1}{m^3}, \frac{1}{m^4}, \frac{1}{m^5}, \frac{1}{m^6}, \ldots \right].$$

To complete the definition $f$, we define $f$ to be periodic with period $\frac{1}{m-1}$. Thus, $f(x + \frac{1}{m-1}) = f(x), \theta \in \mathbb{Z}$. For $m = 2$, $I_0 = [0,1)$ and $f$ has slopes $+2, -4, -8, -16, \ldots$. For $m = 3$, $I_0 = [0, \frac{1}{2})$ and $I_1 = (\frac{1}{2}, 1)$, where $f$ has slopes $+3, -9, +27, \ldots$ on $[0, \frac{1}{2})$, and then is repeated on $[\frac{1}{2}, 1]$. The partition points in $I_h$ are defined by $a_i^{(h)} = \frac{h}{m-1}$,

$$a_i^{(h)} = \frac{h}{m-1} + \sum_{j=1}^{i} \frac{j}{m}, \quad i = 1,2, \ldots.$$  

Thus, on $I_{i+1}^{(h)} = [a_{i}^{(h)}, a_{i+1}^{(h)}], f(x)$ is linear with $f(x) = (-1)^i x^{m+1} + d_i^{(h)}$. It is easy to see that $(m-1)d_i^{(h)} \in \mathbb{Z}, i = 0, 1, 2, \ldots$.

Let $D_N = \left\{ \frac{a}{m} \right\}$, where $p$ is a prime. We assume $(p,m) = (p,m-1) = 1$: since $2$ divides $m$ or $m-1$, it follows that $p > 2$. It is easy to verify that $f: D_N \to D_N$, and thus all points in $D_N$ are eventually periodic. Take for example $m = 2, p = 3, N = 2$.

Then $D_N = \{1/9, 2/9, 4/9, 5/9, 7/9, 8/9\}$. There is only one period: $2/9 \to 4/9 \to 8/9 \to 7/9$, of length $4$, and there are two points which are not periodic. This is in contrast to the case when $[0,1]$ is partitioned into a finite number of intervals when all points of $D_N$ are periodic. See [2].

2. Main Result.

Let $m$ be fixed and consider $f \mid D_N$. The following Theorem gives an explicit upper bound on the number of periodic orbits possessed by $f \mid D_N$. Note that this is shown by obtaining an explicit lower bound for the number of points in any periodic orbit.

Theorem. For fixed $m$, the number of periodic orbits of $f$ that are in $D_N$ is bounded by

$$\left( p^{m'-1} \right) \frac{k-1}{k(1)} \left( 1 + N \frac{\ln p}{\ln m} \right),$$

where $k(1)$ is defined by: $p \mid m^{k(1)} \pm 1, k(1) > 0$ is minimal, and $m'$ is defined by the condition

$$p^{m'} \mid m^{k(1)} \pm 1.$$

Proof: Let $x = \frac{a}{m} \in D_N$ have period $k$. (Note that there is at least one periodic orbit in $D_N$.)

Then $f^k(x) = x$. By the definition of $f$, $f^k(x) = \pm m^{i_k} x + d_k^i$. We shall show that
Notes that this bound is independent of \( x \). Let \( \frac{a}{p^N} \in \mathbb{I}_h^i = \mathbb{I}_h^i, i > 0 \). Then

\[
\frac{h}{m-1} + \sum_{j=1}^{i} \frac{1}{m^j} < \frac{h}{m-1} + \frac{a}{(m-1)p^N} < \frac{h}{m-1} + \sum_{j=1}^{i+1} \frac{1}{m^j} < \frac{h+1}{m-1}.
\]

Note that we cannot have equality, since \( \frac{a}{p^N} \neq a_i^h, h = 0, 1, \ldots, m-2, i = 0, 1, 2, \ldots \). It is easy to see that

\[
\frac{h+1}{m-1} - \frac{a}{(m-1)p^N} \geq \frac{1}{p^{N(m-1)}}.
\]

Thus

\[
\frac{h}{m-1} + \sum_{j=1}^{i} \frac{1}{m^j} < \frac{h+1}{m-1} - \frac{1}{p^{N(m-1)}},
\]

\[
\sum_{j=1}^{i} \frac{1}{m^j} < \frac{1}{m} - \frac{1}{m^j} - \frac{1}{p^{N(m-1)}} < \frac{1}{m-1} - \frac{1}{p^{N(m-1)}},
\]

and \( i < N \frac{\ln p}{\ln m} \). Therefore \( j_1 + 1 < 1 + N \frac{\ln p}{\ln m} \). This is also true if \( i = 0 \); then \( j_1 = 1 \).

Now, \( f^k(x) = \pm m^k \cdot x + d_x = x \), where

\( k' \leq k \leq k(1 + N \frac{\ln p}{\ln m}) \), and \( (m-1)d_x \in \mathbb{Z} \). Thus, since \( p^N \mid m' \pm 1 \),

\[
k' \geq \begin{cases} 
  k(1)p^{N(m-1)} & N \geq m' \\
  k(1) & N < m'
\end{cases}
\]

For more details refer to [1]. Thus \( k > \frac{k(1)p^{N(m-1)}}{1 + N \frac{\ln p}{\ln m}}, N \geq m' \) and \( k > \frac{k(1)}{1 + N \frac{\ln p}{\ln m}}, N < m' \).

(If \( p \) is constant, \( k > \frac{cp^N}{\ln N} \)). It follows that the number of periodic orbits is less than

\[
\left| D_N \right| = \frac{(p-1)p^{N-1}}{(k(1)p^{N(m-1)})/(1 + N \frac{\ln p}{\ln m})}, N \geq m'
\]

\[
= \frac{(p^{m-1})^{p-1}p^{-1}}{k(1)} (1 + N \frac{\ln p}{\ln m}), N \geq m'
\]

If \( N \leq m' \), the bound is \( p^{N-1}(p-1)k(1)(1 + N \frac{\ln p}{\ln m}) \).

**Corollary.** The bound is less than or equal to \( c_i p^{m-1} \frac{p-1}{k(1)} N \ln p \), where \( c_i = 1 + \frac{1}{\ln 2} \).

**Note 1:** Since \( k(1) \geq \frac{\ln(p-1)}{\ln m} \), it is easy to see that this is less than or equal
\[ c_1 \ln_3 4 \cdot p^{m-1} \ln m \cdot N \cdot (p-1). \tag{2.2} \]

Q.E.D.

If \( k(1) = \frac{p-1}{2} \), then the bound is \( 2c_1 p^{m-1} N \ln p \). Note that usually \( m' = 1 \). See \([8, 9]\).

Note that (2.2) is bounded by \( Ap^{\delta}N \).

**Note 2:** A similar result holds for the map \( g: [0,1] \rightarrow [0,1/m] \) which is piecewise linear and continuous on \([0,1/m-1]\) and is then extended as \( f \) is. The function \( g \) is defined on \([0,1/m-1]\) to have slope \(-m, +m, -m, \ldots \) until \( x = 1/m \), then slope \(-m, +m^2, -m^3, +m^4, \ldots \) (if \( m \) is even). (If \( m \) is odd then one starts with slope \(+m, \ldots \).)

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**References**


